JEE–MAIN EXAMINATION – JANUARY 2026

(HELD ON WEDNESDAY 21st JANUARY 2026) **TIME : 3:00 PM TO 6:00 PM**

MATHEMATICS | TEST PAPER WITH SOLUTION

SECTION-A

1. The positive integer n, for which the solutions of the equation

$x(x+2) + (x + 2)(x + 4) + ...+(x + 2n - 2)(x+2n) = \frac{8n}{3}$

are two consecutive even integers, is :-

(1) 3 (2) 6

(3) 12 (4) 9

Ans. (1)

Sol. $x(x+2) + (x+2)(x+4) + ...+ (x+2n-2)(x+2n) = \frac{8n}{3}$

$$\Rightarrow \sum_{r=1}^{n}(x+2r-2)(x+2r) = \frac{8n}{3}$$

$$nx^2 + 2x\sum_{r=1}^{n}(2r-1) + 4\sum_{r=1}^{n}r(r-1) = \frac{8n}{3}$$

$$nx^2 + 2x.n^2 + \frac{4n(n^2-1)}{3} - \frac{8n}{3} = 0$$

$$x^2 + 2nx + \frac{4(n^2-1)}{3} - \frac{8}{3} = 0 <^{\alpha}_{\beta}$$

$$\because |\alpha-\beta| = 2 \Rightarrow \frac{\sqrt{D}}{|a|} = 2 \Rightarrow D = 4$$

$$\Rightarrow 4n^2 - 4\left(4\frac{(n^2-1)}{3} - \frac{8}{3}\right) = 4$$

$$\Rightarrow n^2 - \frac{4n^2}{3} = -3$$

$$\Rightarrow n^2 = 9$$

$$\Rightarrow n = 3$$

2. Let $f : \mathbf{R} \to \mathbf{R}$ be a twice differentiable function such that $f''(x) > 0$ for all $x \in \mathbf{R}$ and $f'(a-1) = 0$, where a is real number. Let $g(x) = f(\tan^2 x - 2\tan x + a)$, $0 < x < \frac{\pi}{2}$.

Consider the following two statements :

(I) g is increasing in $\left(0, \frac{\pi}{4}\right)$

(II) g is decreasing in $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

Then,

(1) Neither (I) nor (II) is True

(2) Only (II) is True

(3) Only (I) is True

(4) Both (I) and (II) are True

Ans. (1)

Sol. $g(x) = f((\tan x - 1)^2 + a - 1)$

$g'(x) = f'((\tan x - 1)^2 + a - 1).2(\tan x - 1)\sec^2 x$

$\because$ $f'(a-1) = 0$ and $f''(x) > 0$

$\therefore f'((\tan x - 1)^2 + a - 1) > 0$

$g'(x) > 0$ if $(\tan x - 1) > 0$

g is increasing in $x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

$g'(x) < 0$ if $\tan x - 1 < 0$

g is decreasing in $x \in \left(0, \frac{\pi}{4}\right)$

3. Let $f(x) = x^3 + x^2 f'(1) + 2x f''(2) + f'''(3)$, $x \in R$. Then the value of $f'(5)$ is :

(1) $\frac{62}{5}$ (2) $\frac{657}{5}$

(3) $\frac{2}{5}$ (4) $\frac{117}{5}$

Ans. (4)

Sol. $f'(x) = 3x^2 + 2xf'(1) + 2f''(2)$

$f''(x) = 6x + 2f'(1)$

$f''(2) = 12 + 2f'(1)$

$\therefore f'(x) = 3x^2 + 2xf'(1) + 2(12 + 2f'(1))$

$f'(x) = 3x^2 + 2(x + 2)f'(1) + 24$

Putting, x = 1

$f'(1) = 3 + 6f'(1) + 24$

$-5f'(1) = 27 \Rightarrow f'(1) = \frac{-27}{5}$

$\therefore f''(2) = 12 + 2\left(\frac{-27}{5}\right) = 12 - \frac{54}{5} = \frac{6}{5}$

$\therefore f'(x) = 3x^2 - \frac{54}{5}x + \frac{12}{5}$

$\therefore f'(5) = 75 - 54 + \frac{12}{5} = \frac{117}{5}$

4. In the line $\alpha x + 4y = \sqrt{7}$, where $\alpha \in R$, touches the ellipse $3x^2 + 4y^2 = 1$ at the point P in the first quadrant, then one of the focal distances of P is :

(1) $\frac{1}{\sqrt{3}} - \frac{1}{2\sqrt{11}}$ (2) $\frac{1}{\sqrt{3}} + \frac{1}{2\sqrt{5}}$

(3) $\frac{1}{\sqrt{3}} - \frac{1}{2\sqrt{5}}$ (4) $\frac{1}{\sqrt{3}} + \frac{1}{2\sqrt{7}}$

Ans. **(4)**

Sol. $\alpha x + 4y - \sqrt{7} = 0$ touches $3x^2 + 4y^2 = 1$

$\therefore c^2 = a^2m^2 + b^2$

$\frac{7}{16} = \frac{1}{3} \times \frac{\alpha^2}{16} + \frac{1}{4} \Rightarrow \alpha = 3, -3$

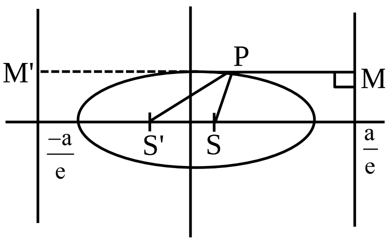
Tangent is $3x + 4y - \sqrt{7} = 0$

Let the point of contact is $P(x_1y_1)$

$\therefore$ Tangent is $3xx_1 + 4yy_1 = 1$

$\therefore \frac{3x_1}{3} = \frac{4y_1}{4} = \frac{1}{\sqrt{7}}$ $\therefore P\left(\frac{1}{\sqrt{7}}, \frac{1}{\sqrt{7}}\right)$

$e = \sqrt{1 - \frac{3}{4}} = \frac{1}{2}$



PS = e (PM)

$= e\left(\frac{a}{e} - \frac{1}{\sqrt{7}}\right)$

$= \frac{1}{2}\left(\frac{2}{\sqrt{3}} - \frac{1}{\sqrt{7}}\right) = \frac{1}{\sqrt{3}} - \frac{1}{2\sqrt{7}}$

$PS' = e\,(PM') = \frac{1}{2}\left(\frac{a}{e} + \frac{1}{\sqrt{7}}\right) = \frac{1}{2}\left(\frac{1}{\sqrt{7}} + \frac{2}{\sqrt{3}}\right)$

$= \frac{1}{\sqrt{3}} + \frac{1}{2\sqrt{7}}$

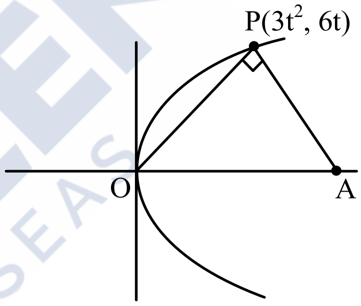
5. Let $y^2 = 12x$ be the parabola with its vertex at O. Let P be a point on the parabola and A be a point on the x-axis such that $\angle OPA = 90°$. Then the locus of the centroid of such triangles OPA is :

(1) $y^2 - 6x + 4 = 0$ (2) $y^2 - 9x + 6 = 0$

(3) $y^2 - 2x + 8 = 0$ (4) $y^2 - 4x + 8 = 0$

Ans. **(3)**

Sol.



$m_{AP} = \frac{-t}{2}$

Equation of AP is

$y - 6t = \frac{-t}{2}(x - 3t^2)$

Put $y = 0 \Rightarrow x = 12 + 3t^2$

$\Rightarrow A(12 + 3t^2, 0)$

Let centroid of Δ OPA be G(h, k)

$\Rightarrow 3h = 0 + 3t^2 + 12 + 3t^2$

$3k = 0 + 6t + 0$

$\Rightarrow t = \frac{k}{2}, \ h = 2t^2 + 4$

$\Rightarrow h = 2\frac{k^2}{4} + 4$

$\Rightarrow$ Locus of (h, k) is

$y^2 = 2x - 8$

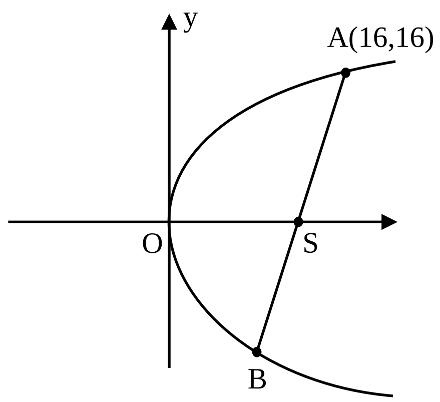
6. Let one end of a focal chord of the parabola $y^2 = 16x$ be (16, 16). If $P(\alpha, \beta)$ divides this focal chord internally in the ratio 5 : 2, then the minimum value of $\alpha + \beta$ is equal to :

(1) 22 (2) 7

(3) 5 (4) 16

Ans. (2)

Sol. $y^2 = 16x$

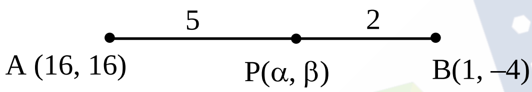


$\because$ parameter of point A is t = 2

$\Rightarrow$ Parameter of point B is $t = -\dfrac{1}{2}$

$\Rightarrow$ Coordinates of B is (1, –4)

Case 1 :

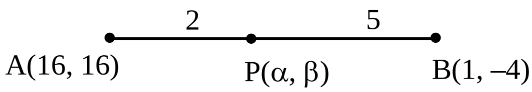


$$\alpha = \frac{5+32}{7} = \frac{37}{7}$$

$$\beta = \frac{-20+32}{7} = \frac{12}{7}$$

$\Rightarrow \alpha + \beta = 7$

Case 2 :



$$\alpha = \frac{2+80}{7}, \qquad \beta = \frac{-8+80}{7}$$

$\alpha + \beta = 22$

So minimum value of $\alpha + \beta = 7$

7. Let the line L pass through the point (–3, 5, 2) and make equal angles with the positive coordinate axes. If the distance of L from the point (–2, r, 1) is $\sqrt{\dfrac{14}{3}}$, then the sum of all possible values of r is :

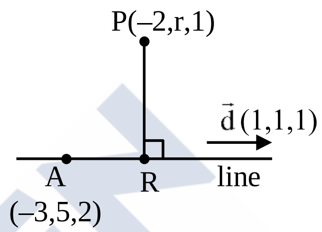
(1) 12 (2) 16

(3) 6 (4) 10

Ans. (4)

Sol. Equation line is : $\dfrac{x+3}{1} = \dfrac{y-5}{1} = \dfrac{z-2}{1} = \lambda$

$\therefore$ General point R on line is $R(\lambda-3, \lambda+5, \lambda+2)$



$\overrightarrow{PR} \equiv (\lambda-1, \lambda+5-r, \lambda+1)$

Now $\overrightarrow{PR} \cdot \vec{d} = 0$

$\Rightarrow (\lambda - 1)1 + (\lambda + 5 - r)1 + (\lambda + 1)1 = 0$

$\Rightarrow 3\lambda - r + 5 = 0$

$\Rightarrow \lambda = \dfrac{r-5}{3}$

$$\therefore R \equiv \left(\frac{r-5}{3} - 3, \frac{r-5}{3} + 5, \frac{r-5}{3} + 2\right)$$

$$R \equiv \left(\frac{r-14}{3}, \frac{r+10}{3}, \frac{r+1}{3}\right)$$

Now

$$PR = \sqrt{\frac{14}{3}} \Rightarrow (PR)^2 = \frac{14}{3}$$

$$\Rightarrow \left(\frac{r-14}{3} + 2\right)^2 + \left(\frac{r+10}{3} - r\right)^2 + \left(\frac{r+1}{3} - 1\right)^2 = \frac{14}{3}$$

$$\Rightarrow \frac{(r-8)^2}{9} + \frac{(10-2r)^2}{9} + \frac{(r-2)^2}{9} = \frac{14}{3}$$

$\Rightarrow (r^2 - 16r + 64) + (100 + 4r^2 - 40r) + (r^2 - 4r + 4) = 42$

$\Rightarrow 6r^2 - 60r + 126 = 0$

$\Rightarrow r^2 - 10r + 21 = 0$

$\Rightarrow r = 3, 7$

sum of possible value of r is = 10

8. Let the line L_1 be parallel to the vector $-3\hat{i}+2\hat{j}+4\hat{k}$ and pass through the point (2, 6, 7) and the line L_2 be parallel to the vector $2\hat{i}+\hat{j}+3\hat{k}$ and pass through the point (4, 3, 5). If the line L_3 is parallel to the vector $-3\hat{i}+5\hat{j}+16\hat{k}$ and intersects the lines L_1 and L_2 at the points C and D, respectively, then $\left|\overrightarrow{CD}\right|^2$ is equal to :

(1) 171 (2) 290

(3) 312 (4) 89

Ans. **(2)**

Sol. $L_1 : \frac{x-2}{-3} = \frac{y-6}{2} = \frac{z-7}{4}$

Point C on L_1: $(-3\lambda_1 + 2, 2\lambda_1 + 6, 4\lambda_1 + 7)$

$L_2 : \frac{x-4}{2} = \frac{y-3}{1} = \frac{z-5}{3}$

Point D on L_2: $(2\lambda_2 + 4, \lambda_2 + 3, 3\lambda_2 + 5)$

Dr's of line L_3 :

$L_3 : \frac{2\lambda_2 + 3\lambda_1 + 2}{-3} = \frac{\lambda_2 - 2\lambda_1 - 3}{5} = \frac{3\lambda_2 - 4\lambda_1 - 2}{16}$

$\lambda_1 = -3, \lambda_2 = 2$

C (11, 0, –5)

D (8, 5, 11)

$\left|\overrightarrow{CD}\right|^2 = 3^2 + 5^2 + 16^2 = 290$

9. Let α and β be the roots of equation $x^2 + 2ax + (3a + 10) = 0$ such that $\alpha < 1 < \beta$. Then the set of all possible values of a is :

(1) $\left(-\infty, \frac{-11}{5}\right) \cup (5, \infty)$ (2) $(-\infty, -2) \cup (5, \infty)$

(3) $(-\infty, -3)$ (4) $\left(-\infty, \frac{-11}{5}\right)$

Ans. **(4)**

Sol. $\because \alpha < 1 < \beta$

$f(1) < 0$

$\Rightarrow 1 + 2a + (3a + 10) < 0$

$\Rightarrow 5a + 11 < 0$

$a < \frac{-11}{5}$

$\therefore a \in \left(-\infty, \frac{-11}{5}\right)$

10. A random variable X takes values 0, 1, 2, 3 with probabilities $\frac{2a+1}{30}, \frac{8a-1}{30}, \frac{4a+1}{30}$, b respectively, where a, b $\in \mathbf{R}$. Let μ and σ respectively be the mean and standard deviation of X such that $\sigma^2 + \mu^2 = 2$. Then $\frac{a}{b}$ is equal to :

(1) 30 (2) 3

(3) 60 (4) 12

Ans. **(3)**

Sol.

x	0	1	2	3
$p(x)$	$\frac{2a+1}{30}$	$\frac{8a-1}{30}$	$\frac{4a+1}{30}$	b

$\sigma^2 = \sum x_i^2 p(x_i) - \mu^2$

$\sigma^2 + \mu^2 = \sum x_i^2 p(x_i)$

$= 0 + 1\left(\frac{8a-1}{30}\right) + 4\left(\frac{4a+1}{30}\right) + 9b$

$\Rightarrow \frac{24a + 270b + 3}{30} = 2$

$24a + 270b = 57$

$8a + 90b = 19$ ….(1)

Also

$\sum p(i) = 1$

$\frac{2a+1}{30} + \frac{8a-1}{30} + \frac{4a+1}{30} + b = 1$

$14a + 30b = 29$ …(2)

Solving (1) & (2)

$a = 2, \quad b = \frac{1}{30}, \ \frac{a}{b} = 60$

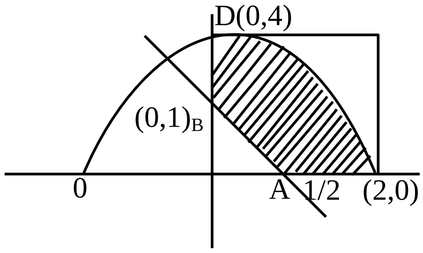
11. If the area of the region $\{(x, y) : 1 - 2x \le y \le 4 - x^2, x \ge 0, y \ge 0\}$ is $\frac{\alpha}{\beta}$, $\alpha, \beta \in N$, gcd $(\alpha, \beta) = 1$, then the value of $(\alpha + \beta)$ is :

(1) 73 (2) 85

(3) 91 (4) 67

Ans. **(1)**

Sol.



Required area $= \frac{2}{3} \times 8 - \frac{1}{2} \times \frac{1}{2} \times 1$

$= \frac{16}{3} - \frac{1}{4} = \frac{61}{12} = \frac{\alpha}{\beta}$

$\Rightarrow \alpha + \beta = 73$

12. Let $a_1, \frac{a_2}{2}, \frac{a_3}{2^2}, \ldots, \frac{a_{10}}{2^9}$ be a G.P. of common ratio $\frac{1}{\sqrt{2}}$. If $a_1 + a_2 + \ldots + a_{10} = 62$, then a_1 is equal to :

(1) $2\left(\sqrt{2}-1\right)$ (2) $2-\sqrt{2}$

(3) $\sqrt{2}-1$ (4) $2(2-\sqrt{2})$

Ans. **(1)**

Sol. $\frac{a_2}{2a_1} = \frac{a_3}{2a_2} = \frac{a_4}{2a_3} = \ldots = \frac{a_{10}}{2a_9} = \frac{1}{\sqrt{2}}$

$\therefore a_1, a_2, a_3, .., a_{10}$ are in G.P. with common ratio $\sqrt{2}$.

$$\sum_{i=1}^{10} a_i = \frac{a_1\left(\left(\sqrt{2}\right)^{10} - 1\right)}{\sqrt{2}-1} = 62$$

$\Rightarrow a_1 = 2\left(\sqrt{2}-1\right)$

13. Let $A = \{x : |x^2 - 10| \le 6\}$ and $B = \{x : |x-2| > 1\}$. Then

(1) $A \cup B = (-\infty, 1] \cup (2, \infty)$

(2) $A - B = [2, 3)$

(3) $A \cap B = [-4, -2] \cup [3, 4]$

(4) $B - A = (-\infty, -4) \cup (-2, 1) \cup (4, \infty)$

Ans. **(4)**

Sol. $|x^2 - 10| \le 6$

$-6 \le x^2 - 10 \le 6$

$4 \le x^2 \le 16$

$A = [-4, -2] \cup [2, 4]$

$|x - 2| > 1$

$B = (-\infty, 1) \cup (3, \infty)$

$A \cup B = (-\infty, 1) \cup [2, \infty)$

$A \cap B = [-4, -2] \cup (3, 4]$

$A - B = [2, 3]$

$B - A = (-\infty, -4) \cup (-2, 1) \cup (4, \infty)$

14. For the matrices $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} -29 & 49 \\ -13 & 18 \end{bmatrix}$, if $(A^{15} + B)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, then among the following which one is true?

(1) $x = 5, y = 7$ (2) $x = 18, y = 11$

(3) $x = 11, y = 2$ (4) $x = 16, y = 3$

Ans. **(3)**

Sol. Here $A^n = \begin{bmatrix} 2n+1 & -4n \\ n & -2n+1 \end{bmatrix}$

$\Rightarrow A^{15} = \begin{bmatrix} 31 & -60 \\ 15 & -29 \end{bmatrix}$

$\Rightarrow A^{15} + B = \begin{bmatrix} 2 & -11 \\ 2 & -11 \end{bmatrix}$

Now $\left(A^{15} + B\right)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 2 & -11 \\ 2 & -11 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow 2x - 11y = 0$

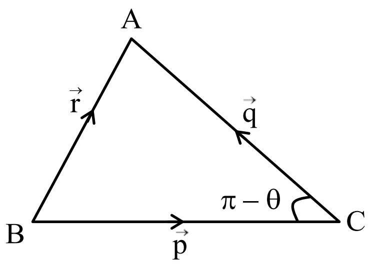
15. For a triangle ABC, let $\vec{p} = \overrightarrow{BC}, \vec{q} = \overrightarrow{CA}$ and $\vec{r} = \overrightarrow{BA}$. If $|\vec{p}| = 2\sqrt{3}, |\vec{q}| = 2$ and $\cos\theta = \frac{1}{\sqrt{3}}$, where θ is the angle between $\vec{P}$ and $\vec{q}$, then $\left|\vec{p} \times (\vec{q} - 3\vec{r})\right|^2 + 3|\vec{r}|^2$ is equal to:

(1) 340 (2) 220

(3) 410 (4) 200

Ans. (4)

Sol.



$\boxed{\vec{p} + \vec{q} = \vec{r}}$

$$\cos(\pi - \theta) = \frac{|\vec{p}|^2 + |\vec{q}|^2 - |\vec{r}|^2}{2|\vec{p}||\vec{q}|}$$

$$\frac{-1}{\sqrt{3}} = \frac{12 + 4 - |\vec{r}|^2}{2 \cdot 2\sqrt{3} \cdot 2}$$

$\boxed{|\vec{r}|^2 = 24}$

$$\therefore \left|\vec{p} \times (\vec{q} - 3\vec{r})\right|^2 + 3|\vec{r}|^2$$

$$= \left|\vec{p} \times (\vec{q} - 3\vec{p} - 3\vec{q})\right|^2 + 72$$

$$= \left|\vec{p} \times (-3\vec{p} - 2\vec{q})\right|^2 + 72$$

$$= \left|-2\vec{p} \times \vec{q}\right|^2 + 72$$

$$= 4|\vec{p}|^2 |\vec{q}|^2 \times \sin^2\theta + 72$$

$$= 4 \cdot 12.4 \cdot \frac{2}{3} + 72$$

$$= 200$$

16. Let y = y(x) be the solution of the differential equation $\sec x \frac{dy}{dx} - 2y = 2 + 3\sin x$, $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $y(0) = -\frac{7}{4}$. Then $y\left(\frac{\pi}{6}\right)$ is equal to:

(1) $-\frac{5}{2}$ (2) $-\frac{5}{4}$

(3) $-3\sqrt{3} - 7$ (4) $-3\sqrt{2} - 7$

Ans. (1)

Sol. $\frac{dy}{dx} - 2y\cos x = 2\cos x + 3\sin x.\cos x$

I.F. = $e^{-2\sin x}$

$$e^{-2\sin x}.y = \int e^{-2\sin x}(3\sin x\cos x + 2\cos x)dx$$

$$y.e^{-2\sin x} = e^{-2\sin x}\left(-\frac{3}{2}\sin x - \frac{7}{4}\right) + C$$

$$\Rightarrow y = -\frac{3}{2}\sin x - \frac{7}{4} + C.e^{2\sin x}$$

$$\because y(0) = -\frac{7}{4} \Rightarrow C = 0$$

$$y\left(\frac{\pi}{6}\right) = \frac{-3}{2}.\frac{1}{2} - \frac{7}{4} = \frac{-5}{2}$$

17. Let A = {2, 3, 5, 7, 9}. Let R be the relation on A defined by x Ry if and only if 2x ≤ 3y. Let ℓ be the number of elements in R, and m be the minimum number of elements required to be added in R to make it a symmetric relation. Then ℓ + m is equal to:

(1) 23 (2) 25

(3) 21 (4) 27

Ans. (2)

Sol. A = {2,3,5,7,9}

$$y \geq \frac{2x}{3}$$

$$\left.\begin{array}{ll} x = 2, & y = 2,3,5,7,9 \\ x = 3, & y = 2,3,5,7,9 \\ x = 5, & y = 5,7,9 \\ x = 7, & y = 5,7,9 \\ x = 9 & y = 7,9 \end{array}\right] \rightarrow \ell = 18$$

to make it symmetric elements to be added are {(5,2), (7,2), (9,2), (5,3),(7,3),(9,3),(9,5)}

$\boxed{m = 7}$

$\therefore \boxed{\ell + m = 25}$

18. If the system of equations

$3x + y + 4z = 3$

$2x + \alpha y - z = -3$

$X + 2y + z = 4$

has no solution, then the value of α is equal to:

(1) 19 (2) 4

(3) 13 (4) 23

Ans. **(1)**

Sol. for no solution $\Delta = 0$

$$\begin{vmatrix} 3 & 1 & 4 \\ 2 & \alpha & -1 \\ 1 & 2 & 1 \end{vmatrix} = 0$$

$\Rightarrow 3(a + 2) + 1(-1 -2) + 4(4 - \alpha) = 0$

$\Rightarrow 19 - \alpha = 0 \Rightarrow \boxed{\alpha = 19}$

& for $\alpha = 19$

$$\Delta_x = \begin{vmatrix} 3 & 1 & 4 \\ -3 & 19 & -1 \\ 4 & 2 & 1 \end{vmatrix} = 3(21) + 1(-1)+4(-82)$$

$\neq 0$

$\therefore$ no solution for $\alpha = 19$

19. Let z be the complex number satisfying $|z-5| \le 3$ and having maximum positive principal argument.

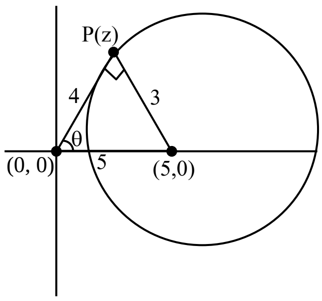
Then $34\left|\frac{5z-12}{5iz+16}\right|^2$ is equal to:

(1) 16 (2) 12

(3) 26 (4) 20

Ans. **(4)**

Sol.



$|z - 5| \le 3$

For arg(z) to be maximum, z lies at P.

$z \equiv (4\cos\theta, 4\sin\theta)$

$$\equiv \left(4\cdot\left(\frac{4}{5}\right), 4\left(\frac{3}{5}\right)\right) = \left(\frac{16}{5}, \frac{12}{5}\right) = \frac{16}{5} + \frac{12i}{5}$$

$$\text{Now, } 34\left|\frac{5z-12}{5iz+16}\right|^2 = 34\left|\frac{(16+12i)-12}{(16i-12)+16}\right|^2$$

$$= 34\left|\frac{4+12i}{16i+4}\right|^2$$

$$= 34\left(\frac{16+144}{256+16}\right) = 34\left(\frac{160}{272}\right) = 20$$

20. The largest $n \in N$, for which 7^n divides 101!, is :

(1) 16 (2) 18

(3) 15 (4) 19

Ans. **(1)**

Sol. Exponent of 7 in 101!

$$= \left[\frac{101}{7}\right] + \left[\frac{101}{7^2}\right] + \left[\frac{101}{7^3}\right] + \ldots\ldots$$

$= 14 + 2 = 16$

SECTION-B

21. Let $[\cdot]$ denote the greatest integer function and $f(x) = \lim\limits_{n\to\infty} \frac{1}{n^3}\sum\limits_{k=1}^{n}\left[\frac{k^2}{3^x}\right]$. Then $12\sum\limits_{j=1}^{\infty} f(j)$ is equal to ________.

Ans. **(2)**

Sol. $$\sum_{k=1}^{n}\left(\frac{k^2}{3^x} - 1\right) < \sum_{k=1}^{n}\left[\frac{k^2}{3^x}\right] \le \sum_{k=1}^{n}\frac{k^2}{3^x}$$

$$\frac{n(n+1)(2n+1)}{6.3^x} < \sum_{k=1}^{n}\left[\frac{k^2}{3^x}\right] \le \frac{n(n+1)(2n+1)}{6.3^x}$$

$$\lim_{n\to\infty}\frac{n(n+1)(2n+1)}{6n^3.3^x} < \lim_{n\to\infty}\frac{1}{n^3}\sum_{k=1}^{n}\left[\frac{k^2}{3^x}\right] \le \lim_{n\to\infty}\frac{n(n+1)(2n+1)}{6.3^x.n^3}$$

$$\frac{1}{3^{x+1}} < \lim_{n\to\infty}\frac{1}{n^3}\sum_{k=1}^{n}\left[\frac{k^2}{3^x}\right] \le \frac{1}{3^{x+1}}$$

$$\Rightarrow f(x) = \frac{1}{3^{x+1}}$$

$$\Rightarrow 12\sum_{j=1}^{\infty} f(j) = 12\sum_{j=1}^{\infty}\frac{1}{3^{j+1}} = 12\left[\frac{1}{9} + \frac{1}{27} + ---\infty\right]$$

$$= 12.\left(\frac{\frac{1}{9}}{1-\frac{1}{3}}\right) = 2$$

22. If $\int_0^1 4\cot^{-1}(1-2x+4x^2)dx = a\tan^{-1}(2) - b\log_e(5)$, where $a, b \in \mathbf{N}$, then $(2a + b)$ is equal to ______.

Ans. **(9)**

Sol. Let $I=\int_0^1 \cot^{-1}(1-2x+4x^2)dx$

$$I=\int_0^1\left(\cot^{-1}(2x-1)-\cot^{-1}(2x)\right)dx \quad \ldots(1)$$

Applying king

$$I=\int_0^1\left(-\cot^{-1}(2x-1)+\cot^{-1}(2x-2)\right)dx \quad \ldots(2)$$

From (1) & (2)

$$2I=\int_0^1\left(\cot^{-1}(2x-2)-\cot^{-1}(2x)\right)dx$$

$$=\int_0^1 \cot^{-1}(2x-2)dx-\int_0^1 \cot^{-1}(2x)dx$$

Applying King

$$=\int_0^1 \cot^{-1}(-2x)dx-\int_0^1 \cot^{-1}(2x)dx$$

$$=\int_0^1 (\pi-\cot^{-1}(2x))dx-\int_0^1 \cot^{-1}(2x)dx$$

$$=\int_0^1 (\pi-2\cot^{-1}(2x))dx$$

$$=\pi-2\int_0^1 (\cot^{-1}2x).1dx$$

By parts

$$=\pi-2\left[\left(x\cot^{-1}2x\right)_0^1+\int_0^1\frac{2x}{1+4x^2}dx\right]$$

Let $1 + 4x^2 = t$

$8xdx = dt$

$$=\pi-2\left[\cot^{-1}2+\frac{1}{4}\int_1^5\frac{dt}{t}\right]$$

$$=\pi-2\cot^{-1}2-\frac{1}{2}\ell n5$$

$$2I=2\tan^{-1}2-\frac{1}{2}\ell n5$$

$\Rightarrow 4I = 4\tan^{-1}2 - \ell n5$

$\therefore\ 2a + b = 8 + 1 = 9$

23. Let the maximum value of $(\sin^{-1}x)^2 + (\cos^{-1}x)^2$ for $x\in\left[-\frac{\sqrt{3}}{2},\frac{1}{\sqrt{2}}\right]$ be $\frac{m}{n}\pi^2$, where gcd (m, n) = 1.

Then m + n is equal to ______.

Ans. **(65)**

Sol. $\left(\sin^{-1}x\right)^2+\left(\cos^{-1}x\right)^2$

$$=\left(\sin^{-1}x+\cos^{-1}x\right)^2-2\sin^{-1}x\cos^{-1}x$$

$$=\frac{\pi^2}{4}-2\left(\sin^{-1}x\right)\left(\frac{\pi}{2}-\sin^{-1}x\right)$$

$$=2\left(\sin^{-1}x-\frac{\pi}{4}\right)^2+\frac{\pi^2}{8} \quad \text{where } \sin^{-1}x\in\left[\frac{-\pi}{3},\frac{\pi}{4}\right]$$

Then max value occurs at $\sin^{-1}x=\frac{-\pi}{3}$

Which is $2\left(\frac{\pi}{3}+\frac{\pi}{4}\right)^2+\frac{\pi^2}{8}=\frac{29\pi^2}{36}$

$\Rightarrow$ m = 29 and n = 36

$\therefore$ m + n = 65

24. If

$$\left(\frac{1}{^{15}C_0}+\frac{1}{^{15}C_1}\right)\left(\frac{1}{^{15}C_1}+\frac{1}{^{15}C_2}\right)\cdots\left(\frac{1}{^{15}C_{12}}+\frac{1}{^{15}C_{13}}\right)=\frac{\alpha^{13}}{^{14}C_0\,^{14}C_1\ldots^{14}C_{12}}$$

, then 30α is equal to _____.

Ans. **(32)**

Sol. $$\prod_{r=0}^{12}\left(\frac{1}{^{15}C_r}+\frac{1}{^{15}C_{r+1}}\right)=\prod_{r=0}^{12}\frac{\frac{16}{r+1}.^{15}C_r}{^{15}C_r.^{15}C_{r+1}}$$

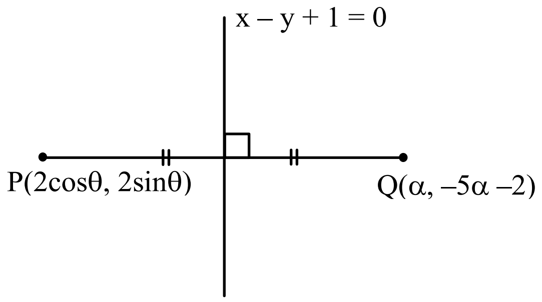
$$=\prod_{r=0}^{12}\frac{16}{(r+1)\cdot\frac{15}{r+1}.^{14}C_r}=\prod_{r=0}^{12}\frac{\left(\frac{16}{15}\right)}{^{14}C_r}$$

$$=\frac{\left(\frac{16}{15}\right)^{13}}{^{14}C_0.^{14}C_1\_\_\_\,^{14}C_{12}}\Rightarrow\alpha=\frac{16}{15}$$

$\Rightarrow 30\alpha = 32$

25. If P is a point on the circle $x^2 + y^2 = 4$, Q is a point on the straight line $5x + y + 2 = 0$ and $x - y + 1 = 0$ is the perpendicular bisector of PQ, then 13 times the sum of abscissa of all such point P is _____.

Ans. (2)



Mid point of PQ lies on $x - y + 1 = 0$

$$\frac{2\cos\theta + \alpha}{2} - \frac{2\sin\theta - 5\alpha - 2}{2} + 1 = 0$$

$2\cos\theta + \alpha - 2\sin\theta + 5\alpha + 2 + 2 = 0$

$\cos\theta - \sin\theta + 3\alpha + 2 = 0 \quad \ldots(1)$

$\because$ Slope of PQ is -1

$$\frac{2\sin\theta + 5\alpha + 2}{2\cos\theta - \alpha} = -1$$

$2\sin\theta + 5\alpha + 2 = -2\cos\theta + \alpha$

$\sin\theta + \cos\theta + 2\alpha + 1 = 0 \quad \ldots(2)$

eliminate α from (1) and (2)

$\Rightarrow \cos\theta + 5\sin\theta = 1,\ \theta \in [0, 2\pi]$

$$\Rightarrow 5 \times 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2\sin^2\frac{\theta}{2}$$

$$\therefore \sin\frac{\theta}{2} = 0 \Rightarrow \cos\theta = 1$$

or

$$\sin\frac{\theta}{2} = 5 \Rightarrow \cos\theta = -\frac{12}{13}$$

Sum of all possible values of abscissa of point P is

$$= 2 \times 1 + 2\left(\frac{-12}{13}\right) = \frac{2}{13}$$

$\therefore$ 13 times sum of all possible values of abscissa of point P is 2.