JEE–MAIN EXAMINATION – JANUARY 2026

(HELD ON THURSDAY 22nd JANUARY 2026) **TIME : 3:00 PM TO 06:00 PM**

MATHEMATICS — TEST PAPER WITH SOLUTION

SECTION-A

1. Let n be the number obtained on rolling a fair die. If the probability that the system

$x - ny + z = 6$

$x + (n-2)y + (n+1)z = 8$

$(n-1)y + z = 1$

Has a unique solution is $\frac{k}{6}$, then the sum of k and all possible values of n is :

(1) 21 (2) 24

(3) 20 (4) 22

Ans. **(4)**

Sol. $x - ny + z = 6$

$x + (n-2)y + (n+1)z = 8$

$(n-1)y + z = 1$

$$\begin{vmatrix} 1 & -n & 1 \\ 1 & (n-2) & n+1 \\ 0 & n-1 & 1 \end{vmatrix} \neq 0$$

$\Rightarrow n^2 - 3n + 2 \neq 0$

$n \neq 1, 2$

for unique solution n = 3, 4, 5, 6

Now

P (probability when system of equations has unique solution) = $\frac{4}{6}$

So k = 4

Now required sum = 4 + (3 + 4 + 5 + 6) = 22

2. If the mean deviation about the median of the numbers k, 2k, 3k, …., 1000k is 500, then k^2 is equal to :

(1) 16 (2) 4

(3) 1 (4) 9

Ans. **(2)**

Sol. $\because$ median $= \frac{1001k}{2} = X_M$

$\therefore$ mean deviation about median $= \frac{\sum |X_i - X_M|}{n}$

$$= \frac{2\left(\frac{k}{2} + \frac{3k}{2} + \frac{5k}{2} + \ldots 500 \text{ terms}\right)}{1000}$$

$$= \frac{2.\frac{k}{2}(500)^2}{1000} = \frac{500k}{2} = 500 \text{ (given)}$$

$\therefore k = 2$

$\therefore k^2 = 4$

3. The number of elements in the relation R = {(x,y): $4x^2 + y^2 < 52$, x, y $\in$ Z} is

(1) 77 (2) 89

(3) 67 (4) 86

Ans. **(1)**

Sol. $4x^2 + y^2 < 52$, $x, y \in Z$

$$\begin{array}{lllll} \downarrow & \downarrow & & & \\ 0 & 0, \pm 1, & \pm 2, \pm 3, \pm 4, \pm 5, \pm 6, \pm 7 & \rightarrow & 1 \times 15 = 15 \\ \pm 1 & & 0, \pm 1, \pm 2, \pm 3 \ldots\ldots\ldots \pm 6 & \rightarrow & 2 \times 13 = 26 \\ \pm 2 & & 0, \pm 1, \pm 2, \pm 3 \ldots\ldots\ldots \pm 5 & \rightarrow & 2 \times 11 = 22 \\ \pm 3 & & 0, \pm 1, \pm 2 \pm 3 & \rightarrow & 2 \times 7 = 14 \end{array}$$

Number of elements = 77

4. Let $S = \left\{ z \in \mathbb{C} : 4z^2 + \bar{z} = 0 \right\}$. Then $\sum_{z \in S} |z|^2$ is equal to :

(1) $\frac{3}{16}$ (2) $\frac{7}{64}$

(3) $\frac{1}{16}$ (4) $\frac{5}{64}$

Ans. **(1)**

Sol. $4z^2 + \bar{z} = 0$

let $z = x + iy$

$4(x + iy)^2 + x - iy = 0$

$4x^2 - 4y^2 + 8xyi + x - iy = 0$

$4x^2 - 4y^2 + x = 0$ & $y(8x - 1) = 0$

$\Rightarrow y = 0$ or $x = \frac{1}{8}$

If $y = 0$, $4x^2 + x = 0$

$x = 0, \frac{-1}{4}$

$\therefore z_1 = 0 + 0.i \quad |z_1|^2 = 0$

$z_2 = 0 - \frac{1}{4}i \qquad |z_2|^2 = \frac{1}{16}$

If $x = \frac{1}{8}$,

$4 \times \frac{1}{64} - 4y^2 + \frac{1}{8} = 0$

$\Rightarrow 4y^2 = \frac{3}{16} \Rightarrow y = \pm\frac{\sqrt{3}}{8}$

$\therefore z_3 = \frac{1}{8} + \frac{\sqrt{3}}{8}i \qquad |z_3|^2 = \frac{1}{64} + \frac{3}{64} = \frac{1}{16}$

$z_4 = \frac{1}{8} - \frac{\sqrt{3}}{8}i \qquad |z_4|^2 = \frac{1}{64} + \frac{3}{64} = \frac{1}{16}$

$\therefore \sum_{i=1}^{n} |z_i|^2 = 0 + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} = \frac{3}{16}$

5. If $\lim_{x\to 0} \frac{e^{(a-1)x} + 2\cos bx + (c-2)e^{-x}}{x\cos x - \log_e(1+x)} = 2$, then $a^2 + b^2 + c^2$ is equal to :

(1) 5 (2) 3

(3) 7 (4) 9

Ans. (3)

Sol. $$\lim_{x\to 0} \frac{\left(1+(a-1)x+\frac{(a-1)^2x^2}{2!}\right)+2\left(1-\frac{b^2x^2}{2!}\right)+(c-2)\left(1-x+\frac{x^2}{2!}\right)}{x\left(1-\frac{x^2}{2!}\right)-\left(x-\frac{x^2}{2}...\right)} = 2$$

$$\lim_{x\to 0} \frac{(1+2+c-2)+x(a-1-c+2)+x^2\left(\frac{(a-1)^2}{2}-b^2+\left(\frac{c-2}{2}\right)\right)}{\frac{x^2}{2}-\frac{x^3}{2!}+...} = 2$$

For which

$\because c + 1 = 0 \Rightarrow c = -1$

$\because a - c = -1 \Rightarrow a = -2$

$\because \frac{(a-1)^2}{2} - b^2 + \left(\frac{c-2}{2}\right) = 1$

$\frac{9}{2} - b^2 - \frac{3}{2} = 1 \Rightarrow b^2 = 2$

$a^2 + b^2 + c^2 = 4 + 2 + 1 = 7$

6. If $y = y(x)$ satisfies the differential equation $16\left(\sqrt{x+9\sqrt{x}}\right)\left(4+\sqrt{9+\sqrt{x}}\right)\cos y\, dy = (1 + 2\sin y)dx$, $x > 0$ and $y(256) = \frac{\pi}{2}$, $y(49) = \alpha$, then $2\sin\alpha$ is equal to :

(1) $2\sqrt{2}-1$ (2) $2\left(\sqrt{2}-1\right)$

(3) $3\left(\sqrt{2}-1\right)$ (4) $\sqrt{2}-1$

Ans. (1)

Sol. $\int \frac{\cos y}{1+2\sin y}dy = \int \frac{dx}{16\left(\sqrt{9\sqrt{x}+x}\right)\left(4+\sqrt{9+\sqrt{x}}\right)}$

$4+\sqrt{9+\sqrt{x}} = t$

$\frac{1}{2\sqrt{9+\sqrt{x}}} \times \frac{dx}{2\sqrt{x}} = 1dx$

$\frac{1}{2}\ell n|1+2\sin y| = \int \frac{4dt}{16t} + C$

$\frac{1}{2}\ell n|1+2\sin y| = \frac{1}{4}\ell n|4+\sqrt{9+\sqrt{x}} + C$

$\frac{1}{2}\ell n(2\sin y+1) = \frac{1}{4}\ell n|4+\sqrt{9+\sqrt{x}}| + C$

Substituting $\left(256, \frac{\pi}{2}\right)$

$\frac{1}{2}\ell n3 = \frac{1}{2}\ell n3 + C \qquad C = 0$

Substituting $(49, \alpha)$

$\frac{1}{2}\ell n(2\sin\alpha+1) = \frac{1}{4}\ell n8$

$\ell n(2\sin\alpha+1) = \frac{1}{2}\ell n8$

$\ell n(2\sin\alpha+1) = \ell n2\sqrt{2}$

$2\sin\alpha + 1 = 2\sqrt{2}$

$2\sin\alpha = 2\sqrt{2}-1$

7. Among the statements

(S1) : If A(5, –1) and B(–2, 3) are two vertices of a triangle, whose orthocentre is (0, 0), then its third vertex is (–4, –7) and

(S2) : If positive numbers 2a, b, c are three consecutive terms of an A.P., then the lines ax + by + c = 0 are concurrent at (2, –2),

(1) Only (S1) is correct

(2) Only (S2) is correct

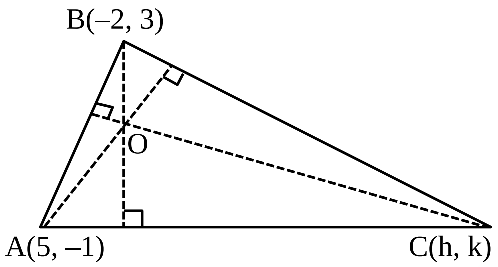
(3) Both are incorrect

(4) Both are correct

Ans. (4)

Sol. Solution of statement-1

$m_{AO}.m_{BC} = -1$



$\Rightarrow 5h - k + 13 = 0$ ....(1)

& $m_{AB}.m_{OC} = -1$

$\Rightarrow 4k = 7h$ ....(2)

$\Rightarrow$ third vertex is (–4, –7)

$\therefore$ Statement 1 is correct.

Solution of statement-2

2a, b, c $\rightarrow$ A.P.

$b = \dfrac{2a+c}{2}$

$\Rightarrow 2a - 2b + c = 0$

$\because$ lines ax + by + c = 0 are concurrent then

$\dfrac{x}{2} = \dfrac{y}{-2} = \dfrac{1}{1}$

x = 2 and y = –2

$\therefore$ Point of concurrency is (2, –2)

$\therefore$ Statement 2 is correct.

8. Let $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{b} = \lambda\hat{j} + 2\hat{k}$, $\lambda \in Z$ be two vectors, Let $\vec{c} = \vec{a} \times \vec{b}$ and $\vec{d}$ be a vector of magnitude 2 in yz-plane. If $|\vec{c}| = \sqrt{53}$, then the maximum possible value of $(\vec{c} \cdot \vec{d})^2$ is equal to :

(1) 26 (2) 104
(3) 208 (4)52

Ans. (3)

Sol. $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$

$\vec{b} = \lambda\hat{j} + 2\hat{k}$; $\lambda \in Z$

$\vec{c} = \vec{a} \times \vec{b} = (-2-\lambda)\hat{i} - 4\hat{j} + 2\lambda\hat{k}$

$|\vec{c}| = \sqrt{53}$

$\Rightarrow 5\lambda^2 + 4\lambda - 33 = 0$

$\lambda = 2.2$ or -3

$\Rightarrow \boxed{\lambda = -3}$

$\vec{c} = \hat{i} - 4\hat{j} - 6\hat{k}$

let $\vec{d} = y\hat{j} + z\hat{k}$

$|\vec{d}| = 2$

$\Rightarrow y^2 + z^2 = 4$

$(\vec{c}.\vec{d})^2 = (4y+6z)^2 \le \left(\sqrt{4^2+6^2} \times \sqrt{y^2+z^2}\right)^2 \le 208$

9. If $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is a solution of the system of equations AX = B, where adj $A = \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$, then |x + y + z| is equal to :

(1) 3 (2) $\dfrac{3}{2}$

(3) 1 (4) 2

Ans. (4)

Sol. $X = A^{-1}B = \left(\dfrac{\text{adj}A}{|A|}\right)B$

$= \pm\dfrac{1}{10}\begin{pmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{pmatrix}\begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix}$

$$= \pm\frac{1}{10}\begin{pmatrix}20\\-10\\10\end{pmatrix} = \pm\begin{pmatrix}2\\-1\\1\end{pmatrix}$$

$\therefore |x + y + z| = 2$

10. Let L be the line $\frac{x+1}{2} = \frac{y+1}{3} = \frac{z+3}{6}$ and let S be the set of all points (a, b, c) on L, whose distance from the line $\frac{x+1}{2} = \frac{y+1}{3} = \frac{z-9}{0}$ along the line L is 7. Then $\sum_{(a,b,c)\in S}(a+b+c)$ is equal to :

(1) 34 (2) 28

(3) 40 (4) 6

Ans. (1)

Sol. M is the point of intersection of L_1 & L_2

$\Rightarrow 2\lambda - 1 = 2\mu - 1,\ 3\lambda - 1 = 3\mu - 1,\ 6\lambda - 3 = 9$

$\Rightarrow \lambda = 2 = \mu$

$\Rightarrow$ M (3, 5, 9)

Now let point P be (2K – 1, 3K – 1, 6K –3) on L_2 such that PM = 7

$\Rightarrow \sqrt{(2K-4)^2 + (3K-6)^2 + (6K-12)^2} = 7$

$\Rightarrow 49K^2 + 196 - 196K = 49$

$\Rightarrow K^2 + 4 - 4K = 1$

$\Rightarrow K^2 - 4K + 3 = 0$

$\Rightarrow K = 1, 3$

So points P & Q are (1, 2, 3) & (5, 8, 15)

So sum of all co-ordinates of P & Q = 34

11. Let P $(10, 2\sqrt{15})$ be a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, whose foci are S and S′. If the length of its latus rectum is 8, then the square of the area of ΔPSS′ is equal to :

(1) 4200 (2) 900

(3) 1462 (4) 2700

Ans. (4)

Sol. $P(10, 2\sqrt{15})$ lies on $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$\therefore \boxed{\frac{100}{a^2} - \frac{60}{b^2} = 1}$ ....(1)

$\because$ length of latus rectum = 8

$\frac{2.b^2}{a} = 8 \Rightarrow \boxed{\frac{b^2}{a} = 4}$ ....(2)

From (1) & (2)

$\frac{100}{a^2} - \frac{60}{4a} = 1$

$400 - 60a = 4a^2$

$4a^2 + 60a - 400 = 0$

$a^2 + 15a - 100 = 0$

a = 5 & –20 (rejected)

$\Rightarrow b = \sqrt{20}$

$\therefore$ Hyperbola is $\boxed{\frac{x^2}{25} - \frac{y^2}{20} = 1}$

$\therefore$ Focal length $S_1S_2 = 2ae = 2.5.\left(\sqrt{1+\frac{4}{5}}\right) = 6\sqrt{5}$

$\therefore$ Area of $\Delta PS_1S_2 = \frac{1}{2}.6\sqrt{5}.2\sqrt{15} = 30\sqrt{3} = A$

$\therefore A^2 = 2700$

12. The area of the region

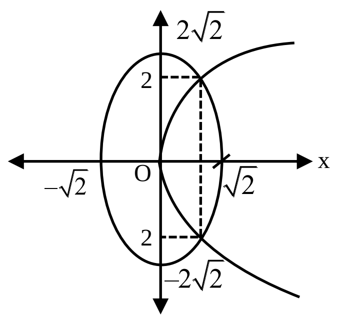
$A = \{(x, y) : 4x^2 + y^2 \le 8$ and $y^2 \le 4x\}$ is :

(1) $\frac{\pi}{2} + 2$ (2) $\pi + \frac{2}{3}$

(3) $\pi + 4$ (4) $\frac{\pi}{2} + \frac{1}{3}$

Ans. (2)

Sol.



$A = \int_0^1 2\sqrt{x}dx + 2\int_1^{\sqrt{2}} \sqrt{8-4x^2}dx$

$= \frac{8}{3}\left(x^{\frac{3}{2}}\right)\Big|_0^1 + 4\int_1^{\sqrt{2}} \sqrt{2-x^2}dx$

$= \frac{8}{3} + 4 \times \frac{1}{2}\left[x\sqrt{2-x^2} + 2\sin^{-1}\left(\frac{x}{\sqrt{2}}\right)\right]\Big|_1^{\sqrt{2}}$

$= \frac{8}{3} + 2\left[2 \times \frac{\pi}{2} - 1 - 2 \times \frac{\pi}{4}\right]$

$= \frac{8}{3} + 2\pi - 2 - \pi = \pi + \frac{2}{3}$ sq. units

13. Let α, β be the roots of the quadratic equation $12x^2 - 20x + 3\lambda = 0$, $\lambda \in \mathbb{Z}$. If $\frac{1}{2} \le |\beta - \alpha| \le \frac{3}{2}$, then the sum of all possible values of λ is :

(1) 6 (2) 1

(3) 3 (4) 4

Ans. (3)

Sol. $\frac{1}{2} \le |\alpha - \beta| \le \frac{3}{2}$

$\frac{1}{4} \le |\alpha - \beta|^2 \le \frac{9}{4}$

$\frac{1}{4} \le (\alpha + \beta)^2 - 4\alpha\beta \le \frac{9}{4}$

$\frac{1}{4} \le \frac{25}{9} - 4 \times \frac{\lambda}{4} \le \frac{9}{4}$

$-\frac{91}{36} \le -\lambda \le \frac{-19}{36}$

$\frac{19}{36} \le \lambda \le \frac{91}{36}$

$\lambda = 1, 2$

Sum = 3

14. Let the domain of the function

$$f(x) = \log_3 \log_5 \left(7 - \log_2 (x^2 - 10x + 85)\right) + \sin^{-1}\left(\left|\frac{3x-7}{17-x}\right|\right)$$

be $(\alpha, \beta]$. Then $\alpha + \beta$ is equal to :

(1) 10 (2) 12

(3) 9 (4) 8

Ans. (3)

Sol. Let $x^2 - 10x + 85 = \lambda$

$\therefore$ Domain for first term

$\lambda > 0$ ...(1)

& $7 - \log_2\lambda > 0 \Rightarrow \lambda < 2^7$ ...(2)

& $\log_5(7 - \log_2\lambda) > 0 \Rightarrow \lambda < 2^6$ ...(3)

$\therefore$ from (1), (2) & (3)

$0 < \lambda < 2^6$

$0 < x^2 - 10x + 85 < 64$

$\Rightarrow x \in (3, 7)$ ...(A)

& domain for second term $-1 \le \frac{3x-7}{x-17} \le 1$

$\Rightarrow x \in [-5, 6]$ ...(B)

From (A) & (B), domain of function will be (3, 6]

$\Rightarrow \alpha = 3, \beta = 6$

$\Rightarrow \alpha + \beta = 9$

15. Let $[\bullet]$ denote the greatest integer function, and let $f(x) = \min\left\{\sqrt{2}x, x^2\right\}$. Let $S = \{x \in (-2, 2)$: the function $g(x) = |x|[x^2]$ is discontinuous at $x\}$.

Then $\sum_{x \in S} f(x)$ equals :

(1) $2 - \sqrt{2}$ (2) $2\sqrt{6} - 3\sqrt{2}$

(3) $1 - \sqrt{2}$ (4) $\sqrt{6} - 2\sqrt{2}$

Ans. (3)

Sol. $g(x) = |x|[x^2]$

points of discontinuity of g(x) in (–2, 2) are

$\left(\pm 1, \pm\sqrt{2}, \pm\sqrt{3}\right)$

$\therefore S = \left\{-1, 1, -\sqrt{2}, \sqrt{2}, -\sqrt{3}, \sqrt{3}\right\}$

$\because f(x) = \min\left\{\sqrt{2}x, x^2\right\}$

$\therefore \sum_{x \in S} f(x) = -\sqrt{2} + 1 - 2 + 2 - \sqrt{6} + \sqrt{6}$

$= 1 - \sqrt{2}$

16. Let S and S' be the foci of the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$ and P (α, β) be a point on the ellipse in the first quadrant. If $(SP)^2 + (S'P)^2 - SP \bullet S'P = 37$, then $\alpha^2 + \beta^2$ is equal to :

(1) 15 (2) 11

(3) 17 (4) 13

Ans. (4)

Sol. $\because$ P lies on ellipse $\Rightarrow \frac{\alpha^2}{25} + \frac{\beta^2}{9} = 1$

$\because PS + PS' = 2a \Rightarrow PS + PS' = 10$

$\therefore (PS)^2 + (PS')^2 - PS.PS' = 37$

$(PS + PS')^2 - 3PS.PS' = 37$

$100 - 3PS.PS' = 37$

$3PS.PS' = 63 \Rightarrow PS.PS' = 21$

$\because$ PS & PS′ are $\left(5 \pm \frac{4}{5} \cdot \alpha\right)$

$\therefore PS.PS' = 25 - \frac{16}{25}\alpha^2 = 21$

$\frac{16}{25}\alpha^2 = 4$

$\alpha = \frac{5}{2} \Rightarrow \alpha^2 = \frac{25}{4}$

$\therefore \beta^2 = \frac{27}{4}$

$\therefore \alpha^2 + \beta^2 = \frac{52}{4} = 13$

17. Let the locus of the mid-point of the chord through the origin O of the parabola $y^2 = 4x$ be the curve S. Let P be any point on S. Then the locus of the point, which internally divides OP in the ratio 3 : 1, is :

(1) $3y^2 = 2x$ (2) $2y^2 = 3x$

(3) $3x^2 = 2y$ (4) $2x^2 = 3y$

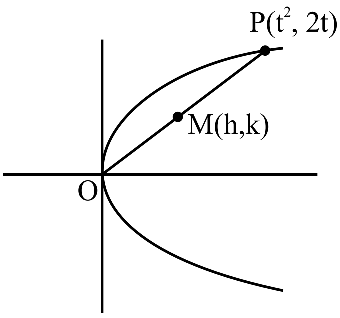
Ans. **(2)**

Sol. $y^2 = 4x$

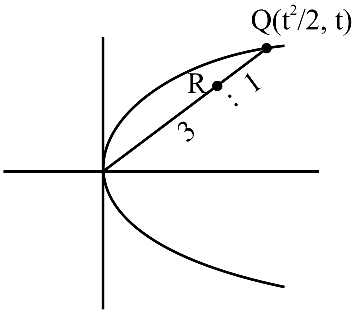
Locus of mid point of OP

$M(h, k) \Rightarrow h = \frac{t^2}{2}, k = t$

$\Rightarrow k^2 = 2h \Rightarrow y^2 = 2x$



$S : y^2 = 2x$



R(h, k)

$\Rightarrow h = \frac{\frac{3t^2}{2}}{4}, k = \frac{3t}{4}$

$t^2 = \frac{8h}{3}, t = \frac{4k}{3}$

$\Rightarrow \frac{16k^2}{9} = \frac{8h}{3} \Rightarrow 2k^2 = 3h$

Locus of R : $2y^2 = 3x$

18. Let $f(x) = [x]^2 - [x + 3] - 3, x \in \mathbb{R}$ where $[\bullet]$ is the greatest integer function. Then

(1) $f(x) > 0$ only for $x \in [4, \infty)$

(2) $f(x) < 0$ only for $x \in [-1, 3)$

(3) $\int_0^2 f(x)dx = -6$

(4) $f(x) = 0$ for finitely many values of x.

Ans. **(2)**

Sol. $f(x) = [x]^2 - [x] - 6 = ([x] + 2)([x] - 3)$

(1) $f(x) > 0 \Rightarrow [x] \in (-\infty, -2) \cup (3, \infty)$

$\Rightarrow x \in (-\infty, -2) \cup [4, \infty)$

(2) $f(x) < 0 \Rightarrow [x] \in (-2, 3)$

$\Rightarrow x \in [-1, 3)$

option (2) is correct

(3) $\int_0^2 f(x)dx = \int_0^1 (0 - 0 - 6)\,dx + \int_1^2 (1 - 1 - 6)\,dx$

$= -6 - 6$

$= -12$

(4) $f(x) = 0 \Rightarrow [x] = 3$ or $[x] = -2$

infinitely many solutions

19. Let f and g be functions satisfying $f(x+y) = f(x)f(y)$, f(1) = 7 and g(x+y) = g(xy), g(1) =1, for all x, y $\in \mathbb{N}$. $\sum_{x=1}^{n}\left(\frac{f(x)}{g(x)}\right) = 19607$, then n is equal to :

(1) 7 (2) 5
(3) 6 (4) 4

Ans. **(2)**

Sol. $f(x+y) = f(x).f(y) \Rightarrow f(x) = a^x$

$(\because f(1) = 7 \Rightarrow = a^1 = 7)$

So $f(x) = 7^x$

Now

$g(x+y) = g(xy)$ (put y = 1)

$\Rightarrow g(x+1) = g(x)$

so $g(1) = g(2) = g(3) = \ldots = g(n) = 1$

Given $\sum_{x=1}^{n}\frac{f(x)}{g(x)} = 19607$

$$\sum_{x=1}^{n}\frac{7^x}{1} = 19607$$

$$\Rightarrow 7\left(\frac{7^n-1}{7-1}\right) = 19607$$

$$7^n - 1 = \frac{6}{7}\times 19607$$

$7^n = 16807 \Rightarrow n = 5$

20. Let C_r denote the coefficient of x^r in the binomial expansion of $(1+x)^n$, $n \in \mathbb{N}$, $0 \le r \le n$.

If $P_n = C_0 - C_1 + \frac{2^2}{3}C_2 - \frac{2^3}{4}C_3 + \ldots + \frac{(-2)^n}{n+1}C_n$,

then the value of $\sum_{n=1}^{25}\frac{1}{P_{2n}}$ equals.

(1) 580 (2) 525
(3) 650 (4) 675

Ans. **(4)**

Sol. $P_n = \sum_{r=0}^{n}\frac{{}^nC_r(-2)^r}{r+1} = \sum_{r=0}^{n}\frac{1}{(n+1)}\,{}^{n+1}C_{r+1}(-2)^r$

$$= \frac{-1}{2(n+1)}\sum_{r=0}^{n}{}^{n+1}C_{r+1}(-2)^{r+1}$$

$$= \frac{-1}{2(n+1)}\left[(1-2)^{n+1}-1\right]$$

$$P_n = \frac{1}{2(n+1)}\left[1-(-1)^{n+1}\right]$$

$$P_{2n} = \frac{1}{2(2n+1)}\left[1-(-1)^{2n+1}\right]$$

$$P_{2n} = \frac{1}{2n+1}$$

$$\sum_{n=1}^{25}\frac{1}{P_{2n}} = \sum_{n=1}^{25}(2n+1)$$

$= 3 + 5 + \ldots + 51$

$= \frac{25}{2}[51+3]$

$= 25 \times 27 = 675$

SECTION-B

21. Let a vector $\vec{a} = \sqrt{2}\hat{i} - \hat{j} + \lambda\hat{k}$, $\lambda > 0$, make an obtuse angle with the vector $\vec{b} = -\lambda^2\hat{i} + 4\sqrt{2}\hat{j} + 4\sqrt{2}\hat{k}$ and an angle θ, $\frac{\pi}{6} < \theta < \frac{\pi}{2}$, with the positive z-axis. If the set of all possible values of λ is $(\alpha, \beta) - \{\gamma\}$, then $\alpha + \beta + \gamma$ is equal to ______.

Ans. **(5)**

Sol. $\frac{\vec{a}.\hat{k}}{|\vec{a}|} = \cos\theta \Rightarrow \frac{\lambda}{\sqrt{3+\lambda^2}} = \cos\theta$

$$\Rightarrow 0 < \frac{\lambda}{\sqrt{3+\lambda^2}} < \frac{\sqrt{3}}{2}$$

$\Rightarrow \lambda > 0 \text{ \& } 4\lambda^2 < 9 + 3\lambda^2 \Rightarrow \lambda^2 < 9$

$\Rightarrow \lambda \in (0, 3)$ ....(1)

$\Rightarrow \vec{a}\cdot\vec{b} < 0 \Rightarrow -\sqrt{2}\lambda^2 - 4\sqrt{2} + 4\sqrt{2}\lambda < 0$

$\Rightarrow \lambda^2 - 4\lambda + 4 > 0 \Rightarrow (\lambda-2)^2 > 0$

$\Rightarrow \lambda \ne 2$ ....(2)

from (1) & (2)

$\lambda \in (0, 3) - \{2\}$

$\therefore \alpha = 0, \beta = 3, \gamma = 2$

$\Rightarrow \alpha + \beta + \gamma = 5$

22. Let [•] be the greatest integer function. If $\alpha = \int_0^{64}\left(x^{1/3} - \left[x^{1/3}\right]\right)dx$, then

$\frac{1}{\pi}\int_0^{\alpha\pi}\left(\frac{\sin^2\theta}{\sin^6\theta + \cos^6\theta}\right)d\theta$ is equal to ____.

Ans. **(36)**

Sol. $\because \int_0^{64} x^{\frac{1}{3}}dx = \frac{3}{4}\cdot\left[x^{\frac{4}{3}}\right]_0^{64} = 192$ &

$$\int_0^{64}\left[x^{1/3}\right]dx = \int_0^1\left[x^{1/3}\right]dx + \int_1^8\left[x^{1/3}\right]dx + \int_8^{27}\left[x^{1/3}\right]dx + \int_{27}^{64}\left[x^{1/3}\right]dx = 156$$

So $\alpha = 192 - 156 = 36$

Now $E = \frac{1}{\pi}\int_0^{36\pi}\frac{\sin^2\theta}{\sin^6\theta + \cos^6\theta}d\theta$

$$= \frac{36}{\pi}\int_0^{\pi}\frac{\sin^2\theta}{\sin^6\theta + \cos^6\theta}d\theta$$

$$\Rightarrow E = \frac{36\cdot 2}{\pi}\int_0^{\pi/2}\frac{\sin^2\theta\, d\theta}{\sin^6\theta + \cos^6\theta}$$

Let $J = \int_0^{\pi/2}\frac{\sin^2\theta}{\sin^6\theta + \cos^6\theta}d\theta$ ....(1)

Applying King

$J = \int_0^{\pi/2}\frac{\cos^2\theta}{\sin^6\theta + \cos^6\theta}d\theta$ ....(2)

Now $2J = \int_0^{\pi/2}\frac{1}{\sin^6\theta + \cos^6\theta}d\theta$ (add (1) & (2))

$$= \int_0^{\pi/2}\frac{\sec^6\theta}{\tan^6\theta + 1}d\theta$$

$$= \int_0^{\infty}\frac{(1+\lambda^2)}{\lambda^4 - \lambda^2 + 1}d\lambda$$

$$= \int_0^{\infty}\frac{1+\frac{1}{\lambda^2}}{\lambda^2 - 1 + \frac{1}{\lambda^2}}d\lambda$$

$= \pi$

$\Rightarrow J = \frac{\pi}{2}$

$\Rightarrow E = \frac{36\cdot 2}{\pi}\times J = 36$

23. Let $\cos(\alpha+\beta) = -\frac{1}{10}$ and $\sin(\alpha-\beta) = \frac{3}{8}$, where

$0 < \alpha < \frac{\pi}{3}$ and $0 < \beta < \frac{\pi}{4}$.

If $\tan 2\alpha = \frac{3(1 - r\sqrt{5})}{\sqrt{11}(s + \sqrt{5})}$, r, s $\in \mathbb{N}$, then r + s is equal to______ .

Ans. **(20)**

Sol. $\tan 2\alpha = \tan[(\alpha+\beta) + (\alpha-\beta)]$

$$\tan 2\alpha = \frac{\tan(\alpha+\beta) + \tan(\alpha-\beta)}{1 - \tan(\alpha+\beta).\tan(\alpha-\beta)}$$

$$\tan 2\alpha = \frac{\left(-\sqrt{99} + \frac{3}{\sqrt{55}}\right)}{1 - \left(\sqrt{99}\right)\left(\frac{3}{\sqrt{55}}\right)}$$

$$\tan 2\alpha = \frac{-3\sqrt{11} + \frac{3}{\sqrt{5}\times\sqrt{11}}}{1 + \frac{9\sqrt{11}}{\sqrt{5}\times\sqrt{11}}}$$

$$\tan 2\alpha = \frac{3\left(1 - 11\sqrt{5}\right)}{\sqrt{11}\left(9 + \sqrt{5}\right)}$$

$r = 11, s = 9$

$r + s = 20$

24. Suppose a, b, c are in A.P. and a^2, $2b^2$, c^2 are in G.P. If $a < b < c$ and $a + b + c = 1$, then $9(a^2 + b^2 + c^2)$ is equal to______ .

Ans. (9)

Sol. $a = b - d,\ c = b + d, \Rightarrow\ b = \frac{1}{3}$

$\Rightarrow 4b^4 = a^2c^2$

$\Rightarrow 4b^4 = [(b - d)(b + d)]^2$

$\Rightarrow \frac{4}{81} = \left(\frac{1}{9} - d^2\right)^2$

$\Rightarrow \left(\frac{1}{9} - d^2\right) = \pm\frac{2}{9}$

$d^2 = 1/3 \Rightarrow\ d = +\frac{1}{\sqrt{3}}$ (as $a < b < c$)

$\therefore\ 9(a^2 + b^2 + c^2)$

$= 9\left[\left(\frac{1}{3} - \frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3} + \frac{1}{\sqrt{3}}\right)^2\right]$

$= 9\left[\frac{1}{3} + \frac{2}{3}\right] = 3 + 6 = 9$

25. Let S be the set of the first 11 natural numbers. Then the number of elements in $A = \{B \subseteq S : n(B) \geq 2$ and the product of all elements of B is even$\}$ is ______ .

Ans. (1979)

Sol. A $\{1,2,3.....11\}$

$\therefore$ $n(B) \geq 2$ & product of all elements in B is even

Case (i) $n(B) = 2 \Rightarrow\ {}^{11}C_2 - {}^{6}C_2$

$n(B) = 3 \Rightarrow {}^{11}C_3 - {}^{6}C_3$

$n(B) = 4 \Rightarrow {}^{11}C_4 - {}^{6}C_4$

$n(B) = 5 \Rightarrow {}^{11}C_5 - {}^{6}C_5$

$n(B) = 6 \Rightarrow {}^{11}C_6 - {}^{6}C_6$

$n(B) = 7 \Rightarrow {}^{11}C_7$

:

:

$n(B) = 11 \Rightarrow {}^{11}C_{11}$

$\therefore$ number of set B $\Rightarrow \sum_{r=2}^{11} {}^{11}C_r - \sum_{r=2}^{6} {}^{6}C_r$

$= 2^{11} - (12) - (2^6 - 7)$

$= 2048 - 64 - 5$

$= 1979$

Alternate Solution :

Total subsets $= 2^{11}$

No. of subsets having odd terms only $= 2^6$

No. of subsets having one term only & also having even terms = 5

Req. ways $= 2^{11} - 2^6 - 5 = 1979$