JEE–MAIN EXAMINATION – APRIL 2025

(HELD ON FRIDAY 04th APRIL 2025) **TIME : 9:00 AM TO 12:00 NOON**

MATHEMATICS | TEST PAPER WITH SOLUTION

SECTION-A

1. Let f, g: $(1, \infty) \to \mathbb{R}$ be defined as $f(x) = \dfrac{2x+3}{5x+2}$ and $g(x) = \dfrac{2-3x}{1-x}$. If the range of the function $fog : [2, 4] \to \mathbb{R}$ is $[\alpha, \beta]$, then $\dfrac{1}{\beta-\alpha}$ is equal to

(1) 68 (2) 29

(3) 2 (4) 56

Ans. (4)

Sol. fog (x) = f(g(x))

$$= f\left(\frac{2-3x}{1-x}\right) = \frac{2\left(\frac{2-3x}{1-x}\right)+3}{5\left(\frac{2-3x}{1-x}\right)+2}$$

$$= \frac{4-6x+3-3x}{10-15x+2-2x} = \left(\frac{7-9x}{12-17x}\right)$$

$$\therefore \left[\begin{array}{l} 12-7x \neq 0 \\ x \neq \dfrac{12}{17} \end{array}\right.$$

$$\left[\begin{array}{l} fog(2) = \dfrac{7-9(2)}{12-17(2)} = \dfrac{-11}{-22} = \dfrac{1}{2} \\ fog(4) = \dfrac{7-9(4)}{12-17(4)} = \dfrac{-29}{-56} = \dfrac{29}{56} \end{array}\right.$$

Range of fog : $[\alpha, \beta] = \left[\dfrac{1}{2}, \dfrac{29}{56}\right]$

$$\therefore (\beta - \alpha) = \frac{29}{56} - \frac{1}{2} = \frac{29-28}{56} = \frac{1}{56}$$

$$\frac{1}{(\beta-\alpha)} = 56$$

2. Consider the sets $A = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 = 25\}$, $B = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 + 9y^2 = 144\}$, $C = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : x^2 + y^2 \leq 4\}$, and $D = A \cap B$. The total number of one-one functions from the set D to the set C is:

(1) 15120 (2) 19320

(3) 17160 (4) 18290

Ans. (3)

Sol. $A : x^2 + y^2 = 25$ ....(1)

$B : \dfrac{x^2}{144} + \dfrac{y^2}{16} = 1$ ....(2)

$C : x^2 + y^2 \leq 4$ ....(3)

Solve (1) & (2)

$x^2 + 9(25 - x^2) = 144$

$-8x^2 = 144 - 225 = -81$

$x = \pm \dfrac{9}{2\sqrt{2}}$

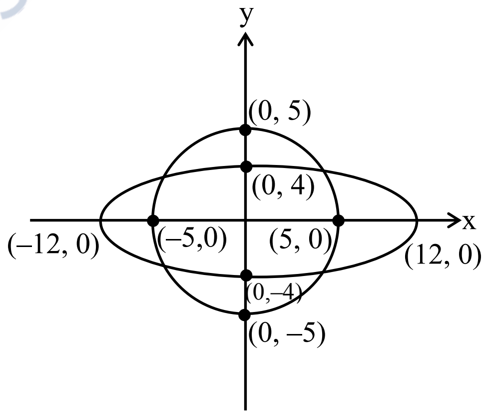
By (1) $\Rightarrow y = \pm\sqrt{25-x^2}$

$= \pm\sqrt{25-\dfrac{81}{8}} = \pm\dfrac{\sqrt{119}}{2\sqrt{2}}$

$\therefore D = A \cap B =$

$$\left\{\left(\frac{9}{2\sqrt{2}}, \frac{\sqrt{119}}{2\sqrt{2}}\right), \left(\frac{9}{2\sqrt{2}}, -\frac{\sqrt{119}}{2\sqrt{2}}\right), \left(\frac{-9}{2\sqrt{2}}, \frac{\sqrt{119}}{2\sqrt{2}}\right), \left(\frac{-9}{2\sqrt{2}}, \frac{-\sqrt{119}}{2\sqrt{2}}\right)\right\}$$

No. of elements in set D = 4



$\because C = \{(x, y) \in Z \times Z : x^2 + y^2 \leq 4\}$

= {(0, 2), (2, 0), (0, –2), (–2, 0), (1, 1), (–1, –1), (1, –1), (–1, 1), (1, 0), (0, 1), (–1, 0), (0, –1), (0, 0)}

No. of elements in set C = 13

Total no. of one-one function from

Set D to sec C $\Rightarrow 13 \times 12 \times 11 \times 10 = 17160$

3. Let A = {1, 6, 11, 16, …} and B = {9, 16, 23, 30, …} be the sets consisting of the first 2025 terms of two arithmetic progressions. Then $n(A \cup B)$ is

(1) 3814 (2) 4027

(3) 3761 (4) 4003

Ans. (3)

Sol. A = {1, 6, 11, 16, 21, 26, 31, 36, 41, 46, 51, 56, 61, 66, 71, 76, 81, 86, 91, ……}

B = {9, 16, 23, 30, 37, 44, 51, 58, 65, 72, 79, 86, 93, 100, ……}

$A \cap B = \{16, 51, 86, \ldots\}$

For set 'A' $\Rightarrow T_{2025} = 1 + (2025 - 1)(5) = 10121$

For set 'B' $\Rightarrow T_{2025} = 9 + (2025 - 1)(7) = 14177$

So, for $(A \cap B) \Rightarrow T_n = 16 + (n-1)(35) \le 10121$

$$(n-1) \le \frac{10121 - 16}{35} = 288.71$$

$n \le 289.71 \Rightarrow n = 289$

$\therefore n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$= 2025 + 2025 - 289 = 3761$

4. For an integer $n \ge 2$, if the arithmetic mean of all coefficients in the binomial expansion of $(x + y)^{2n-3}$ is 16, then the distance of the point $P(2n - 1, n^2 - 4n)$ from the line $x + y = 8$ is:

(1) $\sqrt{2}$ (2) $2\sqrt{2}$

(3) $5\sqrt{2}$ (4) $3\sqrt{2}$

Ans. (4)

Sol. No. of terms in $(x+y)^{(2n-3)} \Rightarrow \begin{bmatrix} (2n-3+1) \\ (2n-2) \end{bmatrix}$

$\therefore$ sum of all coefficients $= 2^{2n-3}$

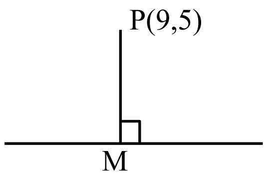
(Put x = y = 1)

$\therefore$ Arithmetic mean of all coefficients

$$= \left(\frac{2^{2n-3}}{2n-2}\right) = 16$$

$\Rightarrow 2^{2n-3} = 2^5(n-1) \Rightarrow n = 5$

$\therefore P(2n-1, n^2 - 4n) = (9, 5)$



$x + y = 8$

$$\therefore PM = \left|\frac{9+5-8}{\sqrt{2}}\right| = \frac{6}{\sqrt{2}} = \frac{3 \times 2}{\sqrt{2}} = 3\sqrt{2}$$

5. The probability, of forming a 12 persons committee from 4 engineers, 2 doctors and 10 professors containing at least 3 engineers and at least 1 doctor, is:

(1) $\frac{129}{182}$ (2) $\frac{103}{182}$

(3) $\frac{17}{26}$ (4) $\frac{19}{26}$

Ans. (1)

Sol. 3 engineering + 1 doctor + 8 Prof $\rightarrow {}^4C_3 . {}^2C_1 . {}^{10}C_8$

= 360

3 engineering + 2 doctors + 7 Prof $\rightarrow {}^4C_3 . {}^2C_2 . {}^{10}C_7$

= 480

4 engineering + 1 doctor + 7 Prof $\rightarrow {}^4C_4 . {}^2C_1 . {}^{10}C_7$

= 240

4 engineering + 2 doctors + 6 Prof $\rightarrow {}^4C_4 . {}^2C_2 . {}^{10}C_6$

= 210

Total = 1290

$$\text{Req. probability} = \frac{1290}{{}^{16}C_{12}} = \frac{1290}{1820} = \frac{129}{182}$$

Ans. (1)

6. Let the shortest distance between the lines $\frac{x-3}{3} = \frac{y-\alpha}{-1} = \frac{z-3}{1}$ and $\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-\beta}{4}$ be $3\sqrt{30}$. Then the positive value of $5\alpha + \beta$ is

(1) 42 (2) 46

(3) 48 (4) 40

Ans. (2)

Sol. $A(3, \alpha, 3)$ & $B(-3, -7, \beta)$

$\overrightarrow{BA} = 6\hat{i} + (\alpha + 7)\hat{j} + (3 - \beta)\hat{k}$

$$\vec{p} \times \vec{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & 1 \\ -3 & 2 & 4 \end{vmatrix}$$

$$\frac{\left|\overrightarrow{BA}.(\vec{p} \times \vec{q})\right|}{|\vec{p} \times \vec{q}|} = 3\sqrt{30}$$

$36 + 15(\alpha + 7) - 3(3 - \beta) = \left(3\sqrt{30}\right)^2$

$36 + 15\alpha + 105 - 9 + 3\beta = 270$

$15\alpha + 3\beta = 138$

$5\alpha + \beta = 46$

7. If $\lim\limits_{x\to 1^+}\dfrac{(x-1)(6+\lambda\cos(x-1))+\mu\sin(1-x)}{(x-1)^3}=-1$,

where $\lambda, \mu \in \mathbb{R}$, then $\lambda + \mu$ is equal to

(1) 18 (2) 20

(3) 19 (4) 17

Ans. **(1)**

Sol. Put x = 1 + h

$$\lim_{h\to 0}\frac{h(6+\lambda\cos h)-\mu\sin h}{h^3}=-1$$

$$\lim_{h\to 0}\frac{h\left(6+\lambda\left(1-\frac{h^2}{2!}\right)\right)-\mu\left(h-\frac{h^3}{3!}\right)}{h^3}=-1$$

$$6+\lambda-\mu=0 \text{ and } -\frac{\lambda}{2}+\frac{\mu}{6}=-1$$

$$\lambda+\mu=18$$

8. Let $f:[0,\infty)\to\mathbb{R}$ be differentiable function such that $f(x)=1-2x+\int_0^x e^{x-t}f(t)\,dt$ for all $x\in[0,\infty)$.

Then the area of the region bounded by $y = f(x)$ and the coordinate axes is

(1) $\sqrt{5}$ (2) $\frac{1}{2}$

(3) $\sqrt{2}$ (4) 2

Ans. **(2)**

Sol. $y=1-2x+e^x\int_0^x e^{-t}f(t)dt$

$$\frac{dy}{dx}=-2+e^{-x}\cdot e^x f(x)+e^x\int_0^x e^{-t}f(t)dt$$

$$\frac{dy}{dx}=-2+y+y+2x-1$$

$$\frac{dy}{dx}-2y=(2x-3)$$

$$ye^{-2x}=\int(2x-3)dx\cdot e^{-2x}$$

$$ye^{-2x}=\frac{-(2x-3)}{2}e^{-2x}+\int e^{-2x}dx$$

$$ye^{-2x}=\frac{-(2x-3)}{2}e^{-2x}-\frac{1}{2}e^{-2x}+c$$

$$f(0)=1\Rightarrow c=1-\frac{3}{2}+\frac{1}{2}=0$$

$$y=-\frac{(2x-3)}{2}-\frac{1}{2}$$

$$y=-x+1$$

$$x+y=1$$

$$\text{area}=\frac{1}{2}(1)(1)=\frac{1}{2}$$

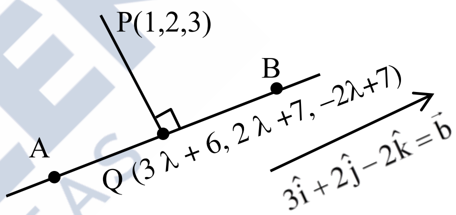
9. Let A and B be two distinct points on the line $L:\frac{x-6}{3}=\frac{y-7}{2}=\frac{z-7}{-2}$. Both A and B are at a distance $2\sqrt{17}$ from the foot of perpendicular drawn from the point (1, 2, 3) on the line L. If O is the origin, then $\overrightarrow{OA}\cdot\overrightarrow{OB}$ is equal to:

(1) 49 (2) 47

(3) 21 (4) 62

Ans. **(2)**

Sol.



$$\overrightarrow{PQ}.\vec{b}=0$$

$$\Rightarrow 3\,(3\lambda+5)+2\,(2\lambda+5)-2\,(-2\lambda+4)$$

$$\Rightarrow 17\lambda=-17\Rightarrow \boxed{\lambda=-1}$$

Q (3,5,9)

Let A $(3\mu+6,\ 2\mu+7,\ -2\mu+7)$

$$(3\mu+3)^2+(2\mu+2)^2+(-2\mu-2)^2=68$$

$$\Rightarrow \mu^2+2\mu-3=0\ \ \mu=-3 \text{ or } \mu=1$$

A (–3, 1, 13) and B (9,9,5)

$$\overrightarrow{OA}.\overrightarrow{OB}=-27+9+65=47$$

10. Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function satisfying $f(0) = 1$ and $f(2x) - f(x) = x$ for all $x\in\mathbb{R}$. If $\lim\limits_{n\to\infty}\left\{f(x)-f\left(\frac{x}{2^n}\right)\right\}=G(x)$, then $\sum\limits_{r=1}^{10}G(r^2)$ is equal to

(1) 540 (2) 385

(3) 420 (4) 215

Ans. **(2)**

Sol. $f(2x) - f(x) = x$

$f(x) - f\left(\frac{x}{2}\right) = \frac{x}{2}$

$f\left(\frac{x}{2}\right) - f\left(\frac{x}{4}\right) = \frac{x}{4}$

$f\left(\frac{x}{4}\right) - f\left(\frac{x}{8}\right) = \frac{x}{8}$

$\vdots$

$f\left(\frac{x}{2^{n-1}}\right) - f\left(\frac{x}{2^n}\right) = \frac{x}{2^n}$

$$f(2x) - f\left(\frac{x}{2^n}\right) = x\left\{\frac{1-\left(\frac{1}{2}\right)^{n-1}}{1-\frac{1}{2}}\right\}$$

$$f(x) - f\left(\frac{x}{2^n}\right) = 2x\left(1-\left(\frac{1}{2}\right)^{n+1}\right)$$

$$f(x) + x - f\left(\frac{x}{2^n}\right) = 2x\left(1-\left(\frac{1}{2}\right)^{n+1}\right)$$

$$\lim_{n\to\infty}\left(f(x) - f\left(\frac{x}{2^n}\right)\right) = \lim_{n\to\infty}\left(2x\left(1-\left(\frac{1}{2}\right)^{n+1}\right) - x\right)$$

$G(x) = x$

$$\sum_{r=1}^{10} G(r^2) = \sum_{r=1}^{10} r^2 = 385$$

11. $1 + 3 + 5^2 + 7 + 9^2 + \ldots$ upto 40 terms is equal to

(1) 43890 (2) 41880

(3) 33980 (4) 40870

Ans. **(2)**

Sol. $(1^2 + 5^2 + 9^2 + \ldots$...upto 20 terms) + (3 + 7 + 11 + ...upto 20 terms)

$$= \sum_{r=1}^{20}(4r-3)^2 + \sum_{r=1}^{20}(4r-1)$$

$$= \sum_{r=1}^{20}(4r-3)^2 + (4r-1)$$

$$= 4\sum_{r=1}^{20}(4r^2 - 5r + 2)$$

$$= 16\sum_{r=1}^{20} r^2 - 20\sum_{r=1}^{20} r + 8\sum_{r=1}^{20} 1 = 41880$$

12. In the expansion of $\left(\sqrt[3]{2} + \frac{1}{\sqrt[3]{3}}\right)^n$, $n \in N$, if the ratio of 15^{th} term from the beginning to the 15^{th} term from the end is $\frac{1}{6}$, then the value of ${}^nC_3$ is:

(1) 4060 (2) 1040

(3) 2300 (4) 4960

Ans. **(3)**

Sol. $T_{r+1} = {}^nC_r (2^{1/3})^{n-r}\left(\frac{1}{3^{1/3}}\right)^r$

$r = 14$

$$T_{15} = {}^nC_{14}(2^{1/3})^{n-14}\left(\frac{1}{3^{1/3}}\right)^{14}$$

$T'_{15} = 15^{th}$ term from last is $(n-13)^{th}$ term from beginning.

$$T'_{15} = {}^nC_{n-14}(2^{1/3})^{14}\left(\frac{1}{3^{1/3}}\right)^{n-14}$$

$$\Rightarrow \frac{T_{15}}{T'_{15}} = \frac{{}^nC_{14}(2^{1/3})^{n-14}\left(\frac{1}{3^{1/3}}\right)^{14}}{{}^nC_{n-14}(2^{1/3})^{14}\left(\frac{1}{3^{1/3}}\right)^{n-14}} = \frac{1}{6}$$

$$= (2^{1/3})^{n-28}(3^{1/3})^{n-28} = \frac{1}{6}$$

$$= 6^{\frac{n-28}{3}} = 6^{-1}$$

$= n = 25$

So, ${}^nC_3 = {}^{25}C_3 = 2300$

13. Considering the principal values of the inverse trigonometric functions,

$\sin^{-1}\left(\frac{\sqrt{3}}{2}x + \frac{1}{2}\sqrt{1-x^2}\right), -\frac{1}{2} < x < \frac{1}{\sqrt{2}}$, is equal to

(1) $\frac{\pi}{4} + \sin^{-1}x$ (2) $\frac{\pi}{6} + \sin^{-1}x$

(3) $\frac{-5\pi}{6} - \sin^{-1}x$ (4) $\frac{5\pi}{6} - \sin^{-1}x$

Ans. **(2)**

Sol. $\sin^{-1}\left(\frac{\sqrt{3}}{2}x+\frac{1}{2}\sqrt{1-x^2}\right)$, $\frac{-1}{2}<x<\frac{1}{\sqrt{2}}$

$\Rightarrow$ Let $\sin^{-1}(x)=\theta$ $\qquad \frac{-\pi}{6}<\theta<\frac{\pi}{4}$

$\Rightarrow x=\sin\theta$, then

$\Rightarrow \sin^{-1}\left(\frac{\sqrt{3}}{2}\sin\theta+\frac{1}{2}\cos\theta\right)$

$\Rightarrow \sin^{-1}\left(\sin\left(\theta+\frac{\pi}{6}\right)\right)=\theta+\frac{\pi}{6}$

$\Rightarrow \sin^{-1}(x)+\frac{\pi}{6}$

14. Consider two vectors $\vec{u}=3\hat{i}-\hat{j}$ and $\vec{v}=2\hat{i}+\hat{j}-\lambda\hat{k}$, $\lambda>0$. The angle between them is given by $\cos^{-1}\left(\frac{\sqrt{5}}{2\sqrt{7}}\right)$. Let $\vec{v}=\vec{v}_1+\vec{v}_2$, where $\vec{v}_1$ is parallel to $\vec{u}$ and $\vec{v}_2$ is perpendicular to $\vec{u}$. Then the value $|\vec{v}_1|^2+|\vec{v}_2|^2$ is equal to

(1) $\frac{23}{2}$ (2) 14

(3) $\frac{25}{2}$ (4) 10

Ans. **(2)**

Sol. $\vec{u}=3\hat{i}-\hat{j}$, $\vec{v}=2\hat{i}+\hat{j}-\lambda\hat{k}$,

$\Rightarrow \frac{\vec{u}.\vec{v}}{|\vec{u}||\vec{v}|}=\cos\theta$

$\Rightarrow \frac{5}{\sqrt{10}\sqrt{5+\lambda^2}}=\frac{\sqrt{5}}{2\sqrt{7}}$

$\Rightarrow \lambda^2=9 \quad \Rightarrow \lambda=3\,(\because \lambda>0)$

$\vec{v}=\vec{v}_1+\vec{v}_2$

$\Rightarrow |\vec{v}|^2=\vec{v}_1^2+\vec{v}_2^2+2\,\vec{v}_1\cdot\vec{v}_2$

$\Rightarrow 14=\vec{v}_1^2+\vec{v}_2^2+0 \qquad (\because \vec{v}_1\perp\vec{v}_2)$

$\Rightarrow |\vec{v}_1^2|+|\vec{v}_1^2|=14$

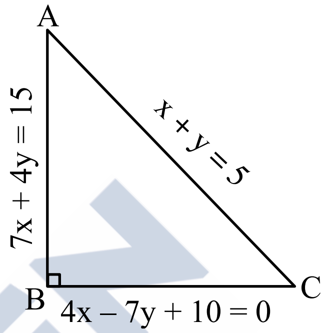
15. Let the three sides of a triangle are on the lines $4x-7y+10=0$, $x+y=5$ and $7x+4y=15$. Then the distance of its orthocentre from the orthocentre of the triangle formed by the lines $x=0$, $y=0$ and $x+y=1$ is

(1) 5 (2) $\sqrt{5}$

(3) $\sqrt{20}$ (4) 20

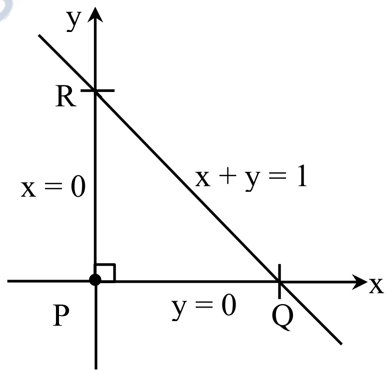
Ans. **(2)**

Sol.



Since triangle is right angle so orthocentre will be at right angle vertex

$7x+4y=15$
$7x+7y+10=0$ } P.O.I B≡(1,2)



distance between P and B = $\sqrt{5}$

16. The value of $\int_{-1}^{1}\frac{\left(1+\sqrt{|x|-x}\right)e^x+\left(\sqrt{|x|-x}\right)e^{-x}}{e^x+e^{-x}}dx$ is equal to

(1) $3-\frac{2\sqrt{2}}{3}$ (2) $2+\frac{2\sqrt{2}}{3}$

(3) $1-\frac{2\sqrt{2}}{3}$ (4) $1+\frac{2\sqrt{2}}{3}$

Ans. **(4)**

Sol. $I=\int_{-1}^{1}\frac{\left(1+\sqrt{|-x|-(-x)}\right)e^{-x}+\left(\sqrt{|-x|-(-x)}\right)e^{-(-x)}}{e^{-x}+e^{-(-x)}}dx$

$\Rightarrow I=\int_{-1}^{1}\frac{\left(1+\sqrt{|x|+x}\right)e^{-x}+\left(\sqrt{|x|+x}\right)e^{x}}{e^{x}+e^{x}}dx$

$\Rightarrow 2I=\int_{-1}^{1}\frac{\left(1+\sqrt{|x|+x}+\sqrt{|x|-x}\right)\left(e^{x}+e^{-x}\right)}{\left(e^{x}+e^{-x}\right)}dx$

$\Rightarrow 2I=\int_{-1}^{1}\left(1+\sqrt{|x|+x}+\sqrt{|x|-x}\right)dx$

$\Rightarrow 2I=2\int_{0}^{1}\left(1+\sqrt{|x|+x}+\sqrt{|x|-x}\right)dx$

$\Rightarrow 2I=2\int_{0}^{1}\left(1+\sqrt{2x}+\sqrt{0}\right)dx$

$\Rightarrow I=\int_{0}^{1}\left(1+\sqrt{2x}\right)dx=\left[x+\frac{2\sqrt{2}}{3}x^{3/2}\right]_{0}^{1}$

$\Rightarrow I=\frac{2\sqrt{2}}{3}+1$

17. The length of the latus-rectum of the ellipse, whose foci are (2, 5) and (2, –3) and eccentricity is $\frac{4}{5}$, is

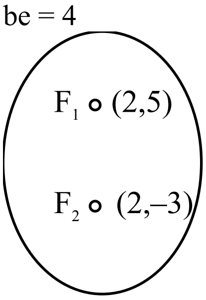
(1) $\frac{6}{5}$ (2) $\frac{50}{3}$

(3) $\frac{10}{3}$ (4) $\frac{18}{5}$

Ans. **(4)**

Sol. $2be = 8$

$be = 4$



$b\left(\frac{4}{5}\right)=4 \Rightarrow b=5$

$\because c^2=b^2-a^2$

$16=25-a^2 \Rightarrow a=3$

L.R. $=\frac{2a^2}{b}=\frac{18}{5}$

Option (4)

18. Consider the equation $x^2 + 4x - n = 0$, where $n \in [20, 100]$ is a natural number. Then the number of all distinct values of n, for which the given equation has integral roots, is equal to

(1) 7 (2) 8

(3) 6 (4) 5

Ans. **(3)**

Sol. $x^2+4x+4=n+4$

$(x+2)^2=n+4$

$x=-2\pm\sqrt{n+4}$

$\because 20\le n\le 100$

$\sqrt{24}\le\sqrt{n+4}\le\sqrt{104}$

$\Rightarrow \sqrt{n+4}\in\{5,6,7,8,9,10\}$

$\therefore$ '6' integral values of 'n' are possible

19. A box contains 10 pens of which 3 are defective. A sample of 2 pens is drawn at random and let X denote the number of defective pens. Then the variance of X is

(1) $\frac{11}{15}$ (2) $\frac{28}{75}$

(3) $\frac{2}{15}$ (4) $\frac{3}{5}$

Ans. **(2)**

Sol.

x	x = 0	x = 1	x = 2
P(x)	$\frac{^{7}C_2}{^{10}C_2}$	$\frac{^{7}C_1\,^{3}C_1}{^{10}C_2}$	$\frac{^{3}C_2}{^{10}C_2}$

$\mu=\Sigma x_i P(x_i)=0+\frac{7}{15}+\frac{2}{15}=\frac{3}{5}$

Variance $(x)=\Sigma P_i(x_i-\mu)^2=\frac{28}{75}$

20. If $10\sin^4\theta + 15\cos^4\theta = 6$, then the value of

$\frac{27\operatorname{cosec}^6\theta+8\sec^6\theta}{16\sec^8\theta}$ is:

(1) $\frac{2}{5}$ (2) $\frac{3}{4}$

(3) $\frac{3}{5}$ (4) $\frac{1}{5}$

Ans. **(1)**

Sol. $10(\sin^2\theta)^2 + 15(1-\sin^2\theta)^2 = 6$

Let $\sin^2\theta = t \Rightarrow 10t^2 + 15(1-t)^2 = 16$

$10t^2 + 15 - 30t + 15t^2 = 6$

$25t^2 - 30t + 9 = 0$

$(5t-3)^2 = 0$

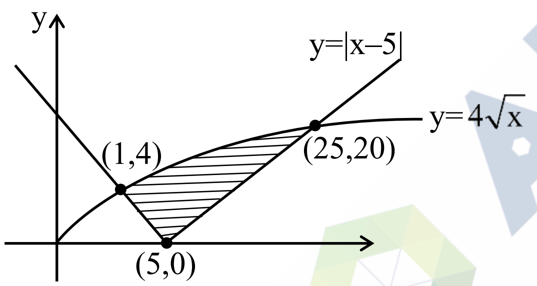
$\sin^2\theta = \frac{3}{5}$ and $\cos^2\theta = \frac{2}{5}$

$$\frac{27\times\frac{125}{27}+8+\frac{125}{8}}{16\left(\frac{5}{2}\right)^4} = \frac{250}{125\times 5} = \frac{2}{5}$$

SECTION-B

21. If the area of the region $\{(x, y) : |x-5| \le y \le 4\sqrt{x}\}$ is A, then 3A is equal to ____.

Ans. **(368)**



Sol.

$$A = \int_1^{25} 4\sqrt{x}\,dx - \frac{1}{2}\times 4\times 4 - \frac{1}{2}\times 20\times 20$$

$$A = \left[\frac{4x^{3/2}}{\frac{3}{2}}\right]_1^{25} - 8 - 200$$

$$A = \frac{8}{3}(125-1) - 208$$

$$A = \frac{368}{3} \Rightarrow 3A = 368$$

22. Let $A = \begin{bmatrix}\cos\theta & 0 & -\sin\theta\\ 0 & 1 & 0\\ \sin\theta & 0 & \cos\theta\end{bmatrix}$. If for some $\theta \in (0, \pi)$, $A^2 = A^T$, then the sum of the diagonal elements of the matrix $(A+I)^3 + (A-I)^3 - 6A$ is equal to ____.

Ans. **(6)**

Sol. $\because$ A is orthogonal matrix

$\therefore A^T = A^{-1}$

$\Rightarrow A^2 = A^{-1}$ $\quad (\because A^2 = A^T)$

$\Rightarrow A^3 = I$

let $B = (A+I)^3 + (A-I)^3 - 6A$

$= 2(A^3 + 3A) - 6A$

$= 2A^3$

$$B = 2I = \begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix}$$

Now sum of diagonal elements = 2 + 2 + 2 = 6

23. Let $A = \{z \in C : |z-2-i| = 3\}$, $B = \{z \in C : \text{Re}(z - iz) = 2\}$ and $S = A \cap B$. Then $\sum_{z\in S}|z|^2$ is equal to ____.

Ans. **(22)**

Sol. Let $z = x + iy$

$A : |z-2-i| = 3$

$|(x-2) + (y-1)i| = 3$

$(x-2)^2 + (y-1)^2 = 9$ $\quad$ ......(1)

$B = \text{Re}(z - iz) = 2$

$\text{Re}((x+y) + i(y-x)) = 2$

$x + y = 2$ $\quad$ ......(2)

On solving (1) and (2) we get

$$x = \frac{3\pm\sqrt{17}}{2},\ y = \frac{1\mp\sqrt{17}}{2}$$

$$\sum_{z\in s}|z|^2 = \frac{1}{4}[2\times 26 + 2\times 18]$$

$$\Rightarrow \frac{88}{4} = 22$$

24. Let C be the circle $x^2 + (y-1)^2 = 2$, E_1 and E_2 be two ellipses whose centres lie at the origin and major axes lie on x-axis and y-axis respectively. Let the straight line $x + y = 3$ touch the curves C, E_1 and E_2 at $P(x_1, y_1)$, $Q(x_2, y_2)$ and $R(x_3, y_3)$ respectively. Given that P is the mid-point of the line segment QR and $PQ = \frac{2\sqrt{2}}{3}$, the value of $9(x_1y_1 + x_2y_2 + x_3y_3)$ is equal to _____.

Ans. (46)

Sol. Let $E_1 : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ (a > b)$

$E_2 : \frac{x^2}{c^2} + \frac{y^2}{d^2} = 1,\ (c < d)$

$C : x^2 + (y-1)^2 = 2$

Equation of tangent at $P(x_1,y_1)$

$xx_1 + y(y_1 - 1) = (y_1 + 1)$

comparing with $x + y = 3$ we get P(1,2)

$\because$ Now parametric equation of $x + y = 3$

$$\frac{(x-1)}{\left(\frac{-1}{\sqrt{2}}\right)} = \frac{(y-2)}{\left(\frac{1}{\sqrt{2}}\right)} = \pm\frac{2\sqrt{2}}{3} \quad \left(\because PQ = \frac{2\sqrt{2}}{3}\right)$$

On solving we get $Q\left(\frac{5}{3}, \frac{4}{3}\right), R\left(\frac{1}{3}, \frac{8}{3}\right)$

So, $9(x_1y_1 + x_2y_2 + x_3y_3)$

$$9\left(2 + \frac{5}{3}\times\frac{4}{3} + \frac{1}{3}\times\frac{8}{3}\right)$$

$\Rightarrow 46$

25. Let m and n be the number of points at which the function $f(x) = \max\{x, x^3, x^5, \ldots, x^{21}\}$, $x \in \mathbb{R}$, is not differentiable and not continuous, respectively. Then m + n is equal to _____.

Ans. (3)

Sol.
$$f(x) = \begin{cases} x, & x < -1 \\ x^{21}, & -1 \le x < 0 \\ x, & 0 \le x < 1 \\ x^{21}, & x \ge 1 \end{cases}$$

f(x) is continuous everywhere.

$\therefore n = 0$

$$f'(x) = \begin{cases} 1, & x < -1 \\ 21x^{20}, & -1 \le x < 0 \\ 1, & 0 < x < 1 \\ 21x^{20}, & x \ge 1 \end{cases}$$

$\therefore$ f(x) is non-differentiable at $x = -1, 0, 1$

$\therefore m = 3$

$m + n = 3$