JEE–MAIN EXAMINATION – APRIL 2025

(HELD ON TUESDAY 08th APRIL 2025) **TIME : 3:00 PM TO 6:00 PM**

MATHEMATICS — TEST PAPER WITH SOLUTION

SECTION-A

1. Let the values of λ for which the shortest distance between the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $\frac{x-\lambda}{3}=\frac{y-4}{4}=\frac{z-5}{5}$ is $\frac{1}{\sqrt{6}}$ be λ_1 and λ_2. Then the radius of the circle passing through the points (0, 0), (λ_1, λ_2) and (λ_2, λ_1) is

(1) $\frac{5\sqrt{2}}{3}$ (2) 4

(3) $\frac{\sqrt{2}}{3}$ (4) 3

Ans. (1)

Sol. $\vec{p}=2\hat{i}+3\hat{j}+4\hat{k}, \vec{q}=3\hat{i}+4\hat{j}+5\hat{k}$

$$\Rightarrow \vec{p}\times\vec{q}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 3 & 4\\ 3 & 4 & 5\end{vmatrix}=-\hat{i}+2\hat{j}-\hat{k}$$

$A \equiv (1, 2, 3)$ $B \equiv (\lambda, 4, 5)$

$$\text{Shortest Distance} = \left|\frac{\overrightarrow{AB}.\left(\vec{P}\times\vec{q}\right)}{\left|\vec{P}\times\vec{q}\right|}\right|$$

$$\frac{1}{\sqrt{6}}=\left|\frac{\left((\lambda-1)\hat{i}+2\hat{j}+2\hat{k}\right).\left(-\hat{i}+2\hat{j}-\hat{k}\right)}{\sqrt{6}}\right|$$

$\Rightarrow |-\lambda+1+4-2|=1 \Rightarrow |\lambda-3|=1$

$\Rightarrow \lambda = 3 \pm 1 = 4, 2$

Radius of circle passing through points (0, 0), (4, 2) & (2, 4)

$$=\frac{abc}{4\Delta}=\frac{\sqrt{20}\times\sqrt{20}\times\sqrt{8}}{4\times\frac{1}{2}\begin{vmatrix}1 & 1 & 1\\ 0 & 4 & 2\\ 0 & 2 & 4\end{vmatrix}}=\frac{20\times 2\sqrt{2}}{2\times 12}$$

$$=\frac{5\sqrt{2}}{3}$$

2. Let α be a solution of $x^2 + x + 1 = 0$, and for some a and b in $\mathbb{R}$, $[4 \quad a \quad b]\begin{bmatrix}1 & 16 & 13\\ -1 & -1 & 2\\ -2 & -14 & -8\end{bmatrix}=[0 \quad 0 \quad 0]$. If $\frac{4}{\alpha^4}+\frac{m}{\alpha^a}+\frac{n}{\alpha^b}=3$, then m + n is equal to ____

(1) 3 (2) 11

(3) 7 (4) 8

Ans. (2)

Sol. $x^2 + x + 1 = 0$

α is root

$\therefore \alpha^2 + \alpha + 1 = 0$

$\Rightarrow \alpha = \omega$ as ω^2 [cube root of unity]

also

$[4 - a - 2b \quad 64 - a - 14b \quad 52 + 2a - 8b]$
$= [0 \quad 0 \quad 0]$

$\therefore a + 2b = 4$

$a + 14b = 64$

$\Rightarrow 12b = 60 \Rightarrow \boxed{b=5}$

$\Rightarrow \boxed{a=-6}$

$$\therefore \frac{4}{\alpha^4}+\frac{m}{\alpha^{-6}}+\frac{n}{\alpha^5}=3$$

$$\Rightarrow \frac{4}{\omega}+\frac{m}{1}+\frac{n}{\omega^2}=3$$

$\Rightarrow 4\omega^2 + m + n\omega = 3$

$$\Rightarrow 4\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)+m+n\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)=3$$

$$\therefore -2+m-\frac{n}{2}=3 \qquad \ldots(1)$$

$$\& \quad \frac{-4\sqrt{3}}{2}+\frac{n\sqrt{3}}{2}=0$$

$\therefore \boxed{n=4}$

$\boxed{m=7}$

$\boxed{\therefore m+n=11}$

3. Let the function $f(x) = \frac{x}{3}+\frac{3}{x}+3,\ x \neq 0$ be strictly increasing in $(-\infty,\alpha_1)U(\alpha_2,\infty)$ and strictly decreasing in $(\alpha_3,\alpha_4)U(\alpha_4,\alpha_5)$. Then $\sum_{i=1}^{5}\alpha_i^2$ is equal to :-

(1) 48 (2) 28

(3) 40 (4) 36

Ans. **(4)**

Sol. $f(x)=\frac{x}{3}+\frac{3}{x}+3,\ x\neq 0$

$f'(x)=\frac{1}{3}-\frac{3}{x^2}=0 \quad \Rightarrow x=\pm 3$

$f'(x)=\frac{x^2-3}{3x^2}$

$f'(x) > 0\ \forall\ (-\infty,-3)\cup(3,\infty) \rightarrow$ increasing

$f'(x) < 0\ \forall\ (-3,0)\cup(0,3) \rightarrow$ decreasing

$\sum_{i=1}^{5}\alpha_i^2=(-3)^2+(3)^2+(-3)^2+(0)^2+(3)^2$

$= 36$

4. If A and B are two events such that P(A) = 0.7, P(B) = 0.4 and $P(A\cap\bar{B})=0.5$, where $\bar{B}$ denotes the complement of B, then $P(B|(A\cup\bar{B}))$ is equal:-

(1) $\frac{1}{4}$ (2) $\frac{1}{2}$

(3) $\frac{1}{6}$ (4) $\frac{1}{3}$

Ans. **(1)**

Sol. $P(A)=\frac{7}{10}$, $P(B)=\frac{4}{10}$

$P(A\cup\bar{B})=\frac{5}{10}$

$P\left(\frac{B}{A\cup\bar{B}}\right)=\frac{P(B\cap(A\cup\bar{B}))}{P(A\cup\bar{B})}$

$=\frac{P((B\cap\bar{B})\cup(B\cap A))}{P(A\cup\bar{B})}=\frac{P(A\cap B)}{P(A\cup\bar{B})}$

$=\frac{P(A)-P(A\cap\bar{B})}{P(A)+P(\bar{B})-P(A\cap\bar{B})}=\frac{\frac{7}{10}-\frac{5}{10}}{\frac{7}{10}+\left(1-\frac{4}{10}\right)-\frac{5}{10}}$

$=\frac{2}{8}=\frac{1}{4}$

5. If $\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+.....\infty=\frac{\pi^4}{90}$,

$\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+.....\infty=\alpha$,

$\frac{1}{2^4}+\frac{1}{4^4}+\frac{1}{6^4}+.....\infty=\beta$,

then $\frac{\alpha}{\beta}$ is equal to

(1) 23 (2) 18

(3) 15 (4) 14

Ans. **(3)**

Sol. If $\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+.....\infty=\frac{\pi^4}{90}$ ....(i)

$\beta=\frac{1}{2^4}+\frac{1}{4^4}+\frac{1}{6^4}+...,$

$=\frac{1}{16}\left[\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+.....\right],$

$=\frac{1}{16}\times\frac{\pi^4}{90}$ using (ii) ....(ii)

$\alpha=\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+.....\infty$

$\left(\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+.....\right)$

$-\left(\frac{1}{2^4}+\frac{1}{4^4}+\frac{1}{6^4}+.....\right)$

$\alpha=\frac{\pi^4}{90}-\frac{1}{16}\times\frac{\pi^4}{90}$ [using (i) and (ii)]

$\alpha=\frac{16-1}{16\times 90}\times\pi^4=\frac{15}{16\times 90}\pi^4=\frac{\pi^4}{96}$

$\therefore \frac{\alpha}{\beta}=\frac{\frac{\pi^4}{96}}{\frac{\pi^4}{16\times 90}}=\frac{16\times 90}{96}=15$

6. The sum of the squares of the roots of $|x+2|^2+|x-2|-2=0$ and the squares of the roots of $x^2-2|x-3|-5=0$, is

(1) 26 (2) 36

(3) 30 (4) 24

Ans. **(2)**

Sol. $|x-2|^2 + 2|x-2| - |x-2| - 2 = 0$

$\Rightarrow (|x-2|+2)(|x-2|-1) = 0$

$\Rightarrow |x-2| = 1$

$\Rightarrow x = 2 \pm 1 = 3, 1$

$\Rightarrow$ sum of square of roots = 9 + 1 = 10

$x^2 - 2|x-3| - 5 = 0$

Case-I $x - 3 \geq 0$

$\Rightarrow x^2 - 2x + 1 = 0$

$\Rightarrow (x-1)^2 = 0$

$\Rightarrow x = 1$

But $x \geq 3$

$\Rightarrow x \in \phi$

Case-II $x - 3 < 0$

$x^2 + 2x - 11 = 0$, $D > 0 \Rightarrow$ Real & distinct roots

$f(x) = x^2 + 2x - 11$

$f(3) > 0$, $\frac{-p}{2a} = -1 < 3$

$\Rightarrow$ both roots < 3, both roots acceptable

Sum of square of roots = $(\alpha+\beta)^2 - 2\alpha\beta$

= 4 + 22 = 26

$\Rightarrow$ Final sum = 10 + 26 = 36

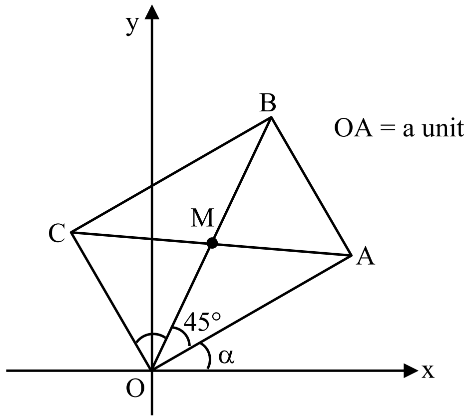
7. Let a be the length of a side of a square OABC with O being the origin. Its side OA makes an acute angle α with the positive x-axis and the equations of its diagonals are $(\sqrt{3}+1)x + (\sqrt{3}-1)y = 0$ and $(\sqrt{3}-1)x - (\sqrt{3}+1)y + 8\sqrt{3} = 0$. Then a^2 is equal to

(1) 48 (2) 32

(3) 16 (4) 24

Ans. (1)

Sol.



Slope of diagonal OB = $\frac{\sqrt{3}+1}{1-\sqrt{3}}$

$= \tan 105°$

$\therefore \alpha = 60°$

$\therefore A(a\cos 60°, a\sin 60°)$

$\therefore A\left(\frac{a}{2}, \frac{\sqrt{3}a}{2}\right)$

A Lies on other diagonal

$\therefore \left(\frac{\sqrt{3}-1}{2}\right)a - \left(\frac{\sqrt{3}+1}{2}\right).\sqrt{3}a + 8\sqrt{3} = 0$

$a\left[\frac{\sqrt{3}-1-3-\sqrt{3}}{2}\right] = -8\sqrt{3}$

$\boxed{a = 4\sqrt{3}}$

$\boxed{\therefore a^2 = 48}$

8. Let f(x) be a positive function and $I_1 = \int_{-\frac{1}{2}}^{1} 2xf(2x(1-2x))\,dx$ and $I_2 = \int_{-1}^{2} f(x(1-x))\,dx$.

Then the value of $\frac{I_2}{I_1}$ is equal to ____

(1) 9 (2) 6

(3) 12 (4) 4

Ans. (4)

Sol. $I_1 = \int_{-\frac{1}{2}}^{1} 2xf(2x(1-2x))dx$

$\Rightarrow 2x = t \Rightarrow 2dx = dt \qquad \Rightarrow I_1 = \frac{1}{2}\int_{-1}^{2} tf(t(1-t))dt$

$\Rightarrow 2I_1 = \int_{-1}^{2}(1-t)f(1-t)(1-(1-t))dt$

$\Rightarrow 2I_1 = \int_{-1}^{2} f(t(1-t)dt - \int_{-1}^{2} tf(t(1-t)dt$

$\Rightarrow 2I_1 = I_2 - 2I_1$

$\Rightarrow 4I_1 = I_2$

$\Rightarrow \frac{I_2}{I_1} = 4$

9. Let $\vec{a} = \hat{i} + 2\hat{j} + \hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} - \hat{k}$. Let $\hat{c}$ be a unit vector in the plane of the vectors $\vec{a}$ and $\vec{b}$ and be perpendicular to $\vec{a}$. Then such a vector $\hat{c}$ is :

(1) $\frac{1}{\sqrt{5}}\left(\hat{j} - 2\hat{k}\right)$ (2) $\frac{1}{\sqrt{3}}\left(-\hat{i} + \hat{j} - \hat{k}\right)$

(3) $\frac{1}{\sqrt{3}}\left(\hat{i} - \hat{j} + \hat{k}\right)$ (4) $\frac{1}{\sqrt{2}}\left(-\hat{i} + \hat{k}\right)$

Ans. (4)

Sol. Let vector $\vec{p}$ in plane of $\vec{a}$ & $\vec{b}$ = $K\left(\vec{a} + \lambda\vec{b}\right)$

$\vec{p} \perp \vec{a} = \vec{p}.\vec{a} = 0$

$\Rightarrow K\left(\vec{a} + \lambda\vec{b}\right).\vec{a} = 0$

$\Rightarrow \vec{a}.\vec{a} + \lambda\vec{b}.\vec{a} = 0$

$\Rightarrow 6 + \lambda(3) = 0$

$\Rightarrow \lambda = -2$

$\Rightarrow \vec{p} = \left(-3\hat{i} + 3\hat{k}\right)$

Unit vector $\rightarrow \pm\frac{\left(-\hat{i} + \hat{k}\right)}{\sqrt{2}}$

10. Let the ellipse $3x^2 + py^2 = 4$ pass through the centre C of the circle $x^2 + y^2 - 2x - 4y - 11 = 0$ of radius r. Let f_1, f_2 be the focal distances of the point C on the ellipse. Then $6f_1f_2 - r$ is equal to

(1) 74 (2) 68

(3) 70 (4) 78

Ans. (3)

Sol. $E : \frac{x^2}{4/3} + \frac{y^2}{4/P} = 1$

Centre of circle (1, 2), radius

$r = \sqrt{1+4+11}$

$\boxed{r = 4}$

$\because$ E pass from centre (1, 2)

$\therefore \frac{3}{4} + P = 1$

$\boxed{P = \frac{1}{4}}$ $\therefore$ vertical ellipse

$e = \sqrt{1 - \frac{4/3}{16}} = \sqrt{1 - \frac{1}{12}} = \sqrt{\frac{11}{12}}$

$\therefore$ Focal distance of C (h, k)

$= b \pm ek$

$F_1 = 4 + \sqrt{\frac{11}{12}} \times 2$

$F_2 = 4 - \sqrt{\frac{11}{12}} \times 2$

$\therefore F_1F_2 = 16 - \frac{11}{3} = \frac{37}{3}$

$\therefore 6F_1F_2 - r = 74 - 4 = 70$

11. The integral $\int_{-1}^{\frac{3}{2}} \left(\left|\pi^2 x \sin(\pi x)\right|\right)$ dx is equal to :

(1) $3 + 2\pi$ (2) $4 + \pi$

(3) $1 + 3\pi$ (4) $2 + 3\pi$

Ans. (3)

Sol. Let, $I = \pi^2 \int_{-1}^{3/2} |x \sin \pi x| dx$

$$= \pi^2 \left\{ \int_{-1}^{1} x \sin \pi x dx - \int_{1}^{3/2} x \sin \pi x dx \right\}$$

$$= \pi^2 \left\{ 2\int_{0}^{1} x \sin \pi x dx - \int_{-1}^{3/2} x \sin \pi x dx \right\}$$

Consider

$$\int x \sin \pi x dx$$

$$-x.\frac{1}{\pi}\cos \pi x + \int 1.\frac{1}{\pi}\cos \pi x dx$$

$$= -\frac{x}{\pi}\cos \pi x + \frac{\sin \pi x}{\pi^2}$$

$$I = \pi^2 \left\{ 2\left(-\frac{x}{\pi}\cos \pi x + \frac{\sin \pi x}{\pi^2}\right)_0^1 - \left(-\frac{x}{\pi}\cos \pi x + \frac{\sin \pi x}{\pi^2}\right)_1^{3/2} \right\}$$

$$= \pi^2 \left\{ \frac{2}{\pi} - \left(-\frac{1}{\pi^2} - \frac{1}{\pi}\right) \right\}$$

$$= \pi^2 \left\{ \frac{3}{\pi} + \frac{1}{\pi^2} \right\}$$

$= 3\pi + 1$

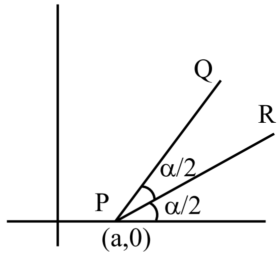
12. A line passing through the point $P(a, \theta)$ makes an acute angle α with the positive x-axis. Let this line be rotated about the point P through an angle $\frac{\alpha}{2}$ in the clock-wise direction. If in the new position, the slope of the line is $2 - \sqrt{3}$ and its distance from the origin is $\frac{1}{\sqrt{2}}$, then the value of $3a^2\tan^2\alpha - 2\sqrt{3}$ is

(1) 4 (2) 6

(3) 5 (4) 8

Ans. (1)

Sol.



$m_{PR} = 2 - \sqrt{3} = \tan 15°$

$\therefore \frac{\alpha}{2} = 15° \quad \Rightarrow \boxed{\alpha = 30°}$

equation of PR :

$y = \tan 15° (x - a)$

$y = (2 - \sqrt{3})(x - a)$

$\perp$ distance from origin $= \frac{1}{\sqrt{2}}$

$$\left|\frac{\sqrt{3}a - 2a}{\sqrt{4 + 3 - 4\sqrt{3} + 1}}\right| = \frac{1}{\sqrt{2}}$$

$$\frac{|a|(2-\sqrt{3})}{2\sqrt{(2-\sqrt{3})}} = \frac{1}{\sqrt{2}}$$

$$|a| = \frac{\sqrt{2}}{\sqrt{2-\sqrt{3}}} = \sqrt{2}(\sqrt{2+\sqrt{3}})$$

$a^2 = 2(2+\sqrt{3})$

$3a^2 \tan^2 \alpha - 2\sqrt{3}$

$3 \times (4 + 2\sqrt{3}).\frac{1}{3} - 2\sqrt{3} = 4$

13. There are 12 points in a plane, no three of which are in the same straight line, except 5 points which are collinear. Then the total number of triangles that can be formed with the vertices at any three of these 12 points is

(1) 230 (2) 220

(3) 200 (4) 210

Ans. (4)

Sol. ${}^{12}C_3 - {}^{5}C_3 = 210$

14. Let $A = \left\{\theta \in [0, 2\pi] : 1 + 10\,\text{Re}\left(\frac{2\cos\theta + i\sin\theta}{\cos\theta - 3i\sin\theta}\right) = 0\right\}$.

Then $\sum_{\theta \in A} \theta^2$ is equal to

(1) $\frac{21}{4}\pi^2$ (2) $8\pi^2$

(3) $\frac{27}{4}\pi^2$ (4) $6\pi^2$

Ans. (1)

Sol. $1 + 10\,\text{Re}\left(\frac{2\cos\theta + i\sin\theta}{\cos\theta - 3i\sin\theta}\right) = 0$

$\therefore\ z + \bar{z} = 2\text{Re}(z)$

$\frac{2\cos\theta + i\sin\theta}{\cos\theta - 3i\sin\theta} + \frac{2\cos\theta - i\sin\theta}{\cos\theta + 3i\sin\theta} = 2 \times \left(\frac{-1}{10}\right)$

$\frac{(2\cos^2\theta - 3\sin^2\theta) + (2\cos^2\theta) - (3\sin^2\theta)}{\cos^2\theta + 9\sin^2\theta} = \frac{-2}{10}$

$\Rightarrow \frac{2\cos^2\theta - 3\sin^2\theta}{\cos^2\theta + 9\sin^2\theta} = \frac{-1}{10}$

$\Rightarrow 20\cos^2\theta - 30\sin^2\theta = -\cos^2\theta - 9\sin^2\theta$

$21\cos^2\theta - 21\sin^2\theta = 0$

$\Rightarrow \cos 2\theta = 0$

$2\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$

$\Rightarrow \sum \theta^2 = \frac{\pi^2}{16} + \frac{9\pi^2}{16} + \frac{25\pi^2}{16} + \frac{49\pi^2}{16} = \frac{84\pi^2}{16} = \frac{21\pi^2}{4}$

15. Let A = {0, 1, 2, 3, 4, 5}. Let R be a relation on A defined by (x, y) ∈ R if and only if max {x, y} ∈ {3, 4}. Then among the statements

(S_1) : The number of elements in R is 18, and

(S_2) : The relation R is symmetric but neither reflexive nor transitive

(1) both are true (2) both are false

(3) only (S_2) is true (4) only (S_1) is true

Ans. (3)

Sol. A = {0,1,2,3,4,5}

R ≡ {(0,3), (3,0), (0,4), (4,0), (1,3), (3,1), (1,4), (4,1), (2,3), (3,2), (2,4), (4,2), (3,3), (3,4), (4,3), (4,4)}

Total 16 elements

Not reflexive as (0,0), ..., (2,2) ∉ R

Symmetric ∵ ∀ all a,b

(a, b) & (b, a) ∈ R

Not transitive ∵ (0,3), (3,1) ∈ R

but (0,1) ∉ R

⇒ Only S_2 correct

16. The number of integral terms in the expansion of $\left(5^{\frac{1}{2}} + 7^{\frac{1}{8}}\right)^{1016}$ is

(1) 127 (2) 130

(3) 129 (4) 128

Ans. (4)

Sol. $T_r = {}^{1016}C_r (5)^{\frac{1016-r}{2}} 7^{\frac{r}{8}}$

$\Rightarrow r = 0, 8, 16, 24, \ldots, 1016$

$1016 = 0 + (n-1)8$

$\Rightarrow n - 1 = \frac{1016}{8} = 127$

So, n = 128.

17. Let f(x) = x – 1 and g(x) = e^x for x ∈ ℝ. If $\frac{dy}{dx} = \left(e^{-2\sqrt{x}} g(f(f(x))) - \frac{y}{\sqrt{x}}\right)$, y(0) = 0, then y(1) is :-

(1) $\frac{1-e^2}{e^4}$ (2) $\frac{2e-1}{e^3}$

(3) $\frac{e-1}{e^4}$ (4) $\frac{1-e^3}{e^4}$

Ans. (3)

Sol. f(x) = x – 1

f(f(x)) = f(x) – 1 = x – 1 – 1 = x – 2

$g(f(f(x))) = e^{x-2}$

$\therefore\ \frac{dy}{dx} = e^{-2\sqrt{x}} \times e^{x-2} - \frac{1}{\sqrt{x}}y$

$\frac{dy}{dx} + \frac{1}{\sqrt{x}}y = e^{x-2\sqrt{x}-2}$ which is L.D.E

I.F. = $e^{\int \frac{dy}{\sqrt{x}}} = e^{2\sqrt{x}}$

Its solution is

$y \times e^{2\sqrt{x}} = \int e^{2\sqrt{x}} \times e^{x-2\sqrt{x}-2} dx + c$

$y \times e^{2\sqrt{x}} = \int e^{x-2} dx + c$

$y \times e^{2\sqrt{x}} = e^{x-2} + c$

Given x = 0, y = 0 ⇒ $0 = e^{-2} + c$; $c = -e^{-2}$

$\therefore\ y \times e^{2\sqrt{x}} = e^{x-2} - e^{-2}$

when x = 1, $y \times e^2 = e^{-1} - e^{-2}$

$y = \frac{e^{-1} - e^{-2}}{e^2} = \frac{\frac{1}{e} - \frac{1}{e^2}}{e^2} = \frac{e^2 - e}{e^5} = \frac{e-1}{e^4}$

Option (1) is correct

18. The value of $\cot^{-1}\left(\frac{\sqrt{1+\tan^2(2)}-1}{\tan(2)}\right) - \cot^{-1}\left(\frac{\sqrt{1+\tan^2\left(\frac{1}{2}\right)}+1}{\tan\left(\frac{1}{2}\right)}\right)$ is equal to

(1) $\pi - \frac{5}{4}$ (2) $\pi - \frac{3}{2}$

(3) $\pi + \frac{3}{2}$ (4) $\pi + \frac{5}{2}$

Ans. (1)

Sol. $\cot^{-1}\left(\frac{|\sec 2|-1}{\tan 2}\right) - \cot^{-1}\left(\frac{\left|\sec\frac{1}{2}\right|+1}{\tan\frac{1}{2}}\right)$

$= \cot^{-1}\left(\frac{-1-\cos 2}{\sin 2}\right) - \cot^{-1}\left(\frac{1+\cos\frac{1}{2}}{\sin\frac{1}{2}}\right)$

$= \pi - \cot^{-1}(\cot 1) - \cot^{-1}\left(\cot\frac{1}{4}\right)$

$= \pi - 1 - \frac{1}{4} = \pi - \frac{5}{4}$

19. Let $A = \begin{bmatrix} 2 & 2+p & 2+p+q \\ 4 & 6+2p & 8+3p+2q \\ 6 & 12+3p & 20+6p+3q \end{bmatrix}$.

If *det* (*adj* (*adj*(3A))) $= 2^m \cdot 3^n$, m, n $\in \mathbb{N}$, then m + n is equal to

(1) 22 (2) 24

(3) 26 (4) 20

Ans. (2)

Sol. $|A| = \begin{vmatrix} 2 & 2+p & 2+p+q \\ 4 & 6+2p & 8+3p+2q \\ 6 & 12+3p & 20+6p+3q \end{vmatrix}$

$C_3 \to C_3 - C_2 - C_1 \times \frac{q}{2}$

Then $C_3 \to C_2 - C_1 x \left(1+\frac{p}{2}\right)$

$\Rightarrow |A| = \begin{vmatrix} 2 & 0 & 0 \\ 4 & 2 & 2+p \\ 6 & 6 & 8+3p \end{vmatrix}$

$\Rightarrow |A| = 2(16 + 6p - 12 - 6p) = 8 = 2^3$

$|adj(adj(3A))| = |3A|^{(3-1)^2} = |3A|^4$

$= (3^3|A|)^4 = (3^3 \times 2^3)^4 = 2^{12} \times 3^{12}$

$\Rightarrow m + n = 24$

20. Given below are two statements :

Statement I :

$$\lim_{x\to 0}\left(\frac{\tan^{-1}x + \log_e\sqrt{\frac{1+x}{1-x}} - 2x}{x^5}\right) = \frac{2}{5}$$

Statement II : $\lim_{x\to 1}\left(x^{\frac{2}{1-x}}\right) = \frac{1}{e^2}$

In the light of the above statements, choose the **correct** answer from the options given below :

(1) Statement I is false but Statement II is true

(2) Statement I is true but Statement II is false

(3) Both Statement I and Statement II are false

(4) Both Statement I and Statement II are true

Ans. (4)

Sol. $\lim\limits_{x\to 0}\dfrac{\tan^{-1}x+\frac{1}{2}\left[\ell n(1+x)-\ell n(1-x)\right]-2x}{x^5}$

$=\lim\limits_{x\to 0}\dfrac{\left(x-\frac{x^3}{3}+\frac{x^5}{5}....\right)+\frac{1}{2}\left[x-\frac{x^2}{2}+\frac{x^3}{3}...-\left(-x-\frac{x^2}{2}-\frac{x^3}{3}...\right)\right]-2x}{x^5}$

$=\lim\limits_{x\to 0}\dfrac{2x+\frac{2x^5}{5}....-2x}{x^5}=\dfrac{2}{5}$

$\lim\limits_{x\to 1}x^{\frac{2}{(1-x)}}=e^{\lim\limits_{x\to 1}\left(\frac{2}{1-x}\right)(x-1)}=e^{-2}$

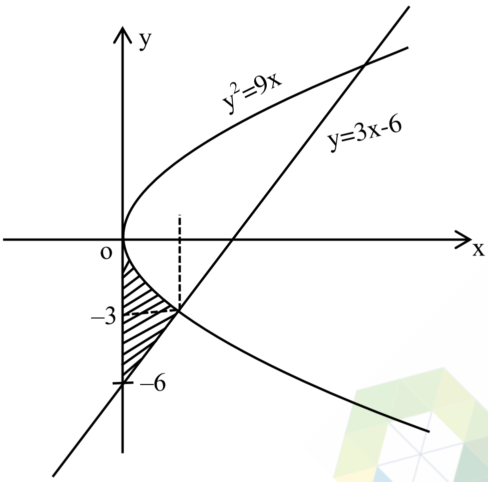
$\Rightarrow$ Both statements correct

SECTION-B

21. Let the area of the bounded region $\{(x, y) : 0 \le 9x \le y^2, y \ge 3x - 6\}$ be A. Then 6A is equal to ______

Ans. (15)

Sol. $0 \le 9x \le y^2$ & $y \ge 3x - 6$



$A = \text{Required Area} = \left[\int_0^1\left(-3\sqrt{x}\right)dx-\int_0^1(3x-6)dx\right]$

$A=-3\left(\dfrac{x^{\frac{3}{2}}}{\frac{3}{2}}\right)\Bigg|_0^1-\left(\dfrac{3x^2}{2}-6x\right)\Bigg|_0^1$

$A=-2[1-0]\left[\dfrac{3}{2}-6\right]$

$A=-2-\dfrac{3}{2}+6=\dfrac{5}{2}$ Sq. unit

$\therefore\ 6A=6\times\dfrac{5}{2}=15$

22. Let the domain of the function $f(x)=\cos^{-1}\left(\dfrac{4x+5}{3x-7}\right)$ be $[\alpha, \beta]$ and the domain of $g(x) = \log_2(2 - 6\log_{27}(2x + 5))$ be (γ, δ). Then $|7(\alpha + \beta) + 4(\gamma + \delta)|$ is equal to ______

Ans. (96)

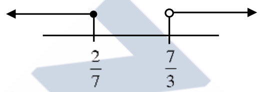
Sol. $f(x)=\cos^{-1}\left(\dfrac{4x+5}{3x-7}\right)$

$\Rightarrow -1\le\left(\dfrac{4x+5}{3x-7}\right)\le 1$

$\left(\dfrac{4x+5}{3x-7}\right)\ge -1$

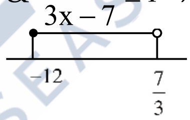
$\dfrac{4x+5+3x-7}{3x-7}\ge 0$

$\Rightarrow \dfrac{7x-2}{3x-7}\ge 0$



$x\in\left(-\infty,\dfrac{2}{7}\right]\cup\left(\dfrac{7}{3},\infty\right)$

& $\dfrac{4x+5}{3x-7}\le 1\Rightarrow\dfrac{x+12}{3x-7}\le 0$



$\therefore$ Domain of f(x) is

$\left[-12,\dfrac{2}{7}\right]$ $\boxed{\alpha=-12,\beta=\dfrac{2}{7}}$

$g(x) = \log_2(2-6\log_{27}(2x + 5))$

Domain

$2 - 6\log_{27}(2x + 5) > 0$

$\Rightarrow\ 6\log_{27}(2x + 5) < 2$

$\Rightarrow\ \log_{27}(2x + 5) < \dfrac{1}{3}$

$\Rightarrow\ 2x + 5 < 3$

$\Rightarrow\ x < -1$

& $2x + 5 > 0 \Rightarrow x > -\dfrac{5}{2}$

Domain is $x\in\left(-\dfrac{5}{2},-1\right)$

$\boxed{\gamma=-\dfrac{5}{2},\delta=-1}$

$|7(\alpha + \beta) + 4(\gamma + \delta)| = |7(-12+\dfrac{2}{7})+4(-\dfrac{5}{2}-1)|$

$|-82 - 14| = 96$

23. Let the area of the triangle formed by the lines $x + 2 = y - 1 = z$, $\frac{x-3}{5} = \frac{y}{-1} = \frac{z-1}{1}$ and $\frac{x}{-3} = \frac{y-3}{3} = \frac{z-2}{1}$ be A. Then A^2 is equal to ______

Ans. **(56)**

Sol. $L_1 : x + 2 = y - 1 = z = \ell$

$$L_2 : \frac{x-3}{5} = \frac{y}{-1} = \frac{z-1}{1} = m$$

$$L_3 : \frac{x}{-3} = \frac{y-3}{5} = \frac{z-2}{1} = n$$

Point of intersection of L_1 and L_2

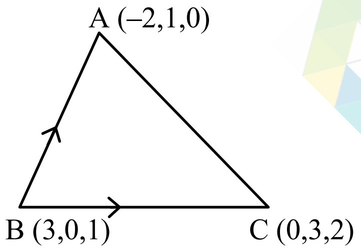
$$\left.\begin{array}{r}\ell - 2 = 5m + 3\\ \ell + 1 = -m\\ \ell = m + 1\end{array}\right\} \ell = 0, m = -1 \quad A(-2, 1, 0)$$

Point of intersection of L_2 and L_3

$$\left.\begin{array}{r}5m + 3 = -3n\\ -m = 3n + 3\\ m + 1 = n + 2\end{array}\right\} m = 0, n = -1, B(3, 0, 1)$$

Point of intersection L_3 and L_4

$$\left.\begin{array}{r}-3n = \ell - 2\\ 3n + 3 = \ell + 1\\ n + 2 = \ell\end{array}\right\} \ell = 2,\ n = 0, C(0, 3, 2)$$



$$Ar(\Delta ABC) = \left|\frac{1}{2}\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -5 & 1 & -1\\ -3 & 3 & 1\end{vmatrix}\right|$$

$$A = \frac{1}{2}|\hat{i}(4) - \hat{j}(-8) + \hat{k}(-12)|$$

$$A = \frac{1}{2}\sqrt{16 + 64 + 144} = \sqrt{56}$$

$A^2 = 56$

24. The product of the last two digits of $(1919)^{1919}$ is ______

Ans. **(63)**

Sol. $(1919)^{1919} = (1920 - 1)^{1919}$

$= {}^{1919}C_0(1920)^{1919} - {}^{1919}C_1(1920)^{1918} + \ldots$

$+ {}^{1919}C_{1918}(1920)^1 - {}^{1919}C_{1919}$

$= 100\lambda + 1919 \times 1920 - 1$

$= 100\lambda + 3684480 - 1$

$= 100\lambda + \ldots\ldots\ldots. 79$ (last two digit)

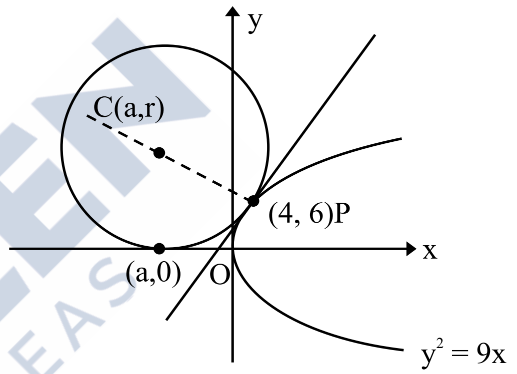
$\Rightarrow$ Number having last two digit 79

$\therefore$ Product of last two digit 63

25. Let r be the radius of the circle, which touches x-axis at point (a, 0), $a < 0$ and the parabola $y^2 = 9x$ at the point (4, 6). Then r is equal to ______

Ans. **(30)**

Sol.



$(x - a)^2 + (y - r)^2 = r^2$

$(4 - a)^2 + (6 - r)^2 = r^2$

$16 + a^2 - 8a + 36 + r^2 - 12r = r^2$

$a^2 - 8a - 12r + 52 = 0$

Tangent to parabola at (4, 6) is

$6.4 = 9.\left(\frac{x+4}{2}\right)$ i.e. $3x - 4y + 12 = 0$

This is also tangent to the circle

$\therefore$ CP = r

$$\frac{3a - 4r + 12}{5} = \pm r$$

$$3a + 12 = 4r \pm 5r\begin{cases} ar \\ -r \end{cases} \qquad \ldots..(1)$$

equation of circle is

$(x - a)^2 + (y - r)^2 = r^2$

satsty P(4, 6) $\Rightarrow a^2 - 8a - 12r + 52 = 0$ .....(2)

From equation (1)

If $a + 4 = 3r$ then $a = +6$ (rejected)

If $3a + 12 = -r$ then $a = -14$ and $r = 30$