JEE–MAIN EXAMINATION – JANUARY 2025

(HELD ON THURSDAY 23rd JANUARY 2025) **TIME : 9:00 AM TO 12:00 NOON**

MATHEMATICS — TEST PAPER WITH SOLUTION

SECTION-A

1. The value of $\int_{e^2}^{e^4} \frac{1}{x}\left(\frac{e^{\left((\log_e x)^2+1\right)^{-1}}}{e^{\left((\log_e x)^2+1\right)^{-1}}+e^{\left((6-\log_e x)^2+1\right)^{-1}}}\right)dx$ is

(1) $\log_e 2$ (2) 2

(3) 1 (4) e^2

Ans. **(3)**

Sol. Let $\ln x = t \Rightarrow \frac{dx}{x} = dt$

$$I = \int_2^4 \frac{e^{\frac{1}{1+t^2}}}{e^{\frac{1}{1+t^2}}+e^{\frac{1}{1+(6-t)^2}}}dt$$

$$I = \int_2^4 \frac{e^{\frac{1}{1+(6-t)^2}}}{e^{\frac{1}{1+(6-t)^2}}+e^{\frac{1}{1+t^2}}}dt$$

$$2I = \int_2^4 dt = (t)_2^4 = 4-2 = 2$$

$I = 1$

2. Let $I(x) = \int \frac{dx}{(x-11)^{\frac{11}{13}}(x+15)^{\frac{15}{13}}}$.

If $I(37) - I(24) = \frac{1}{4}\left(\frac{1}{b^{\frac{1}{13}}}-\frac{1}{c^{\frac{1}{13}}}\right)$, b, c $\in \mathbb{N}$, then

3(b + c) is equal to

(1) 40 (2) 39

(3) 22 (4) 26

Ans. **(2)**

Sol. $I(x) = \int \frac{dx}{(x-11)^{\frac{11}{13}}(x+15)^{\frac{15}{13}}}$

Put $\frac{x-11}{x+15} = t \Rightarrow \frac{26}{(x+5)^2}dx = dt$

$$I(x) = \frac{1}{26}\int \frac{dt}{t^{11/13}} = \frac{1}{26}\cdot\frac{t^{2/13}}{2/13}$$

$$I(x) = \frac{1}{4}\left(\frac{x-11}{x+15}\right)^{2/13} + C$$

$$I(37) - I(24) = \frac{1}{4}\left(\frac{26}{52}\right)^{2/13} - \frac{1}{4}\left(\frac{13}{39}\right)^{2/13}$$

$$= \frac{1}{4}\left(\frac{1}{2^{2/13}}-\frac{1}{3^{2/13}}\right)$$

$$= \frac{1}{4}\left(\frac{1}{4^{1/13}}-\frac{1}{9^{1/13}}\right)$$

$\therefore$ b = 4, c = 9

3(b + c) = 39

3. If the function

$$f(x) = \begin{cases} \frac{2}{x}\{\sin(k_1+1)x+\sin(k_2-1)x\} & , x<0 \\ 4 & , x=0 \\ \frac{2}{x}\log_e\left(\frac{2+k_1x}{2+k_2x}\right) & , x>0 \end{cases}$$

is continuous at x = 0, then $k_1^2 + k_2^2$ is equal to

(1) 8 (2) 20

(3) 5 (4) 10

Ans. **(4)**

Sol. $\lim_{x\to 0^-} \frac{2}{x}\{\sin(k_1+1)x+\sin(k_2-1)x\} = 4$

$\Rightarrow 2(k_1 + 1) + 2(k_2 - 1) = 4$

$\Rightarrow k_1 + k_2 = 2$

$\Rightarrow \lim_{x\to 0^+} \frac{2}{x}\ln\left(\frac{2+k_1x}{2+k_2x}\right) = 4$

$\Rightarrow \lim_{x\to 0^+} \frac{1}{x}\ln\left(1+\frac{(k_1-k_2)x}{2+k_2x}\right) = 2$

$\Rightarrow \frac{k_1-k_2}{2} = 2$

$\Rightarrow k_1 - k_2 = 4$

$\therefore k_1 = 3, k_2 = -1$

$k_1^2 + k_2^2 = 9+1 = 10$

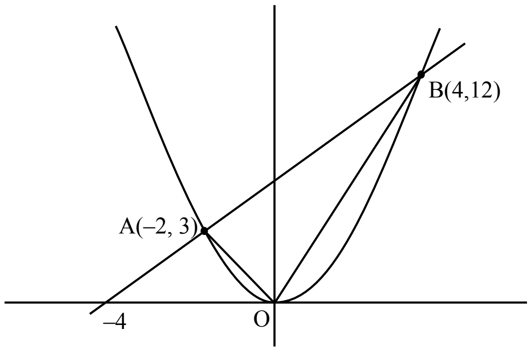
4. If the line $3x - 2y + 12 = 0$ intersects the parabola $4y = 3x^2$ at the points A and B, then at the vertex of the parabola, the line segment AB subtends an angle equal to

(1) $\tan^{-1}\left(\frac{11}{9}\right)$ (2) $\frac{\pi}{2} - \tan^{-1}\left(\frac{3}{2}\right)$

(3) $\tan^{-1}\left(\frac{4}{5}\right)$ (4) $\tan^{-1}\left(\frac{9}{7}\right)$

Ans. **(4)**

Sol.



$3x - 2y + 12 = 0$

$4y = 3x^2$

$\therefore 2(3x + 12) = 3x^2$

$\Rightarrow x^2 - 2x - 8 = 0$

$\Rightarrow x = -2, 4$

$m_{OA} = -3/2, m_{OB} = 3$

$$\tan\theta = \left|\frac{\frac{-3}{2} - 3}{1 - \frac{9}{2}}\right| = \frac{9}{7}$$

$\theta = \tan^{-1}\left(\frac{9}{7}\right)$ (angle will be acute)

5. Let a curve $y = f(x)$ pass through the points $(0, 5)$ and $(\log_e 2, k)$. If the curve satisfies the differential equation $2(3 + y)e^{2x}dx - (7 + e^{2x})dy = 0$, then k is equal to

(1) 16 (2) 8

(3) 32 (4) 4

Ans. **(2)**

Sol. $\frac{dy}{dx} = \frac{2(3+y).e^{2x}}{7+e^{2x}}$

$\frac{dy}{dx} - \frac{2y.e^{2x}}{7+e^{2x}} = \frac{6.e^{2x}}{7+e^{2x}}$

$\text{I.F.} = e^{-\int \frac{2e^{2x}}{7+e^{2x}}dx} = \frac{1}{7+e^{2x}}$

$\therefore y.\frac{1}{7+e^{2x}} = \int \frac{6e^{2x}}{\left(7+3^{2x}\right)^2}dx$

$\frac{y}{7+e^{2x}} = \frac{-3}{7+e^{2x}} + C$

$(0, 5) \Rightarrow \frac{5}{8} = \frac{-3}{8} + C \Rightarrow C = 1$

$\therefore y = -3 + 7 + e^{2x}$

$y = e^{2x} + 4$

$\therefore k = 8$

6. Let $f(x) = \log_e x$ and $g(x) = \frac{x^4 - 2x^3 + 3x^2 - 2x + 2}{2x^2 - 2x + 1}$. Then the domain of fog is

(1) $\mathbb{R}$ (2) $(0, \infty)$

(3) $[0, \infty)$ (4) $[1, \infty)$

Ans. **(1)**

Sol. $f(x) = \ln x$

$g(x) = \frac{x^4 - 2x^3 + 3x^2 - 2x + 2}{2x^2 - 2x + 1}$

$D_g \in R$

$D_f \in (0, \infty)$

For $D_{fog} \Rightarrow g(x) > 0$

$\frac{x^4 - 2x^3 + 3x^2 - 2x + 2}{2x^2 - 2x + 1} > 0$

$\Rightarrow x^4 - 2x^3 + 3x^2 - 2x + 2 > 0$

Clearly $x < 0$ satisfies which are included in option (1) only.

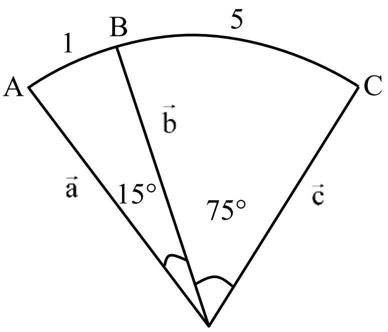
7. Let the arc AC of a circle subtend a right angle at the centre O. If the point B on the arc AC, divides the arc AC such that $\frac{\text{length of arc AB}}{\text{length of arc BC}} = \frac{1}{5}$, and $\overrightarrow{OC} = \alpha\overrightarrow{OA} + \beta\overrightarrow{OB}$, then $\alpha = \sqrt{2}\left(\sqrt{3}-1\right)\beta$ is equal to

(1) $2-\sqrt{3}$ (2) $2\sqrt{3}$

(3) $5\sqrt{3}$ (4) $2+\sqrt{3}$

Ans. (1)

Sol.



$\vec{c} = \alpha\vec{a} + \beta\vec{b}$ ....(1)

$\vec{a}.\vec{c} = \alpha\vec{a}.\vec{a} + \beta\vec{b}.\vec{a}$

$0 = \alpha + \beta \cos 15°$ ....(2)

(1) $\Rightarrow \vec{b}.\vec{c} = \alpha\vec{a}.\vec{b} + \beta\vec{b}.\vec{b}$

$\Rightarrow \cos 75° = \alpha \cos 15° + \beta$ ....(3)

(2) & (3) $\Rightarrow \cos 75° = -\beta\cos^2 15° + \beta$

$$\beta = \frac{\cos 75^\circ}{\sin^2 15^\circ} = \frac{1}{\sin 15^\circ} = \frac{2\sqrt{2}}{\sqrt{3}-1}$$

$$(2) \Rightarrow \alpha = \frac{-\cos 15^\circ}{\sin 15^\circ} = \frac{-(\sqrt{3}+1)}{(\sqrt{3}-1)}$$

$$\therefore \vec{c} = \frac{-(\sqrt{3}+1)}{(\sqrt{3}-1)}\vec{a} + \left(\frac{2\sqrt{2}}{\sqrt{3}-1}\right)\vec{b}$$

Now

$$\alpha + \sqrt{2}(\sqrt{3}-1)\beta = \frac{-(\sqrt{3}+1)}{(\sqrt{3}-1)} + \frac{\sqrt{2}(\sqrt{3}-1).2\sqrt{2}}{\sqrt{3}-1}$$

$$= \frac{-(\sqrt{3}+1)^2}{2} + 4$$

$$= \frac{-3-1-2\sqrt{3}+8}{2}$$

$$= 2-\sqrt{3}$$

8. If the first term of an A.P. is 3 and the sum of its first four terms is equal to one-fifth of the sum of the next four terms, then the sum of the first 20 terms is equal to

(1) –1200 (2) –1080

(3) –1020 (4) –120

Ans. (2)

Sol. $a = 3$

$S_4 = \frac{1}{5}(S_8 - S_4)$

$\Rightarrow 5S_4 = S_8 - S_4$

$\Rightarrow 6S_4 = S_8$

$\Rightarrow 6.\frac{4}{2}[2\times3+(4-1)\times d]$

$= \frac{8}{2}[2\times3+(8-1)d]$

$\Rightarrow 12(6 + 3d) = 4(6 + 7d)$

$\Rightarrow 18 + 9d = 6 + 7d$

$\Rightarrow d = -6$

$S_{20} = \frac{20}{2}[2\times3+(20-1)(-6)]$

$= 10\,[6 - 114]$

$= -1080$

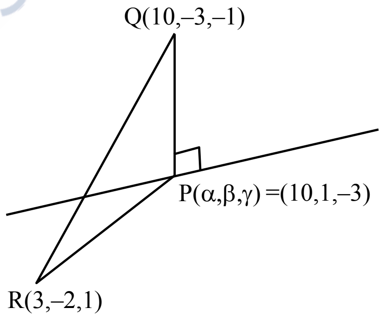
9. Let P be the foot of the perpendicular from the point Q(10, –3, –1) on the line $\frac{x-3}{7} = \frac{y-2}{-1} = \frac{z+1}{-2}$. Then the area of the right angled triangle PQR, where R is the point (3, –2, 1), is

(1) $9\sqrt{15}$ (2) $\sqrt{30}$

(3) $8\sqrt{15}$ (4) $3\sqrt{30}$

Ans. (4)

Sol.



$\frac{x-3}{7} = \frac{y-2}{-1} = \frac{z+1}{-2} = \lambda$

$\Rightarrow 7\lambda + 3, -\lambda + 2, -2\lambda - 1$

dr's of QP $\Rightarrow$

$7\lambda - 7, -\lambda + 5, -2\lambda$

Now

$(7\lambda - 7).7 - (-\lambda + 5) + (2\lambda).2 = 0$

$54\lambda - 54 = 0 \Rightarrow \lambda = 1$

$\therefore P = (10, 1, -3)$

$\overrightarrow{PQ} = -4\hat{j} + 2\hat{k}$

$\overrightarrow{PR} = -7\hat{i} - 3\hat{j} + 4\hat{k}$

$$\text{Area} = \left\| \frac{1}{2} \begin{vmatrix} i & j & k \\ 0 & -4 & 2 \\ -7 & -3 & 4 \end{vmatrix} \right\| = 3\sqrt{30}$$

10. Let $\left|\frac{\bar{z}-i}{2\bar{z}+i}\right|=\frac{1}{3}$, $z \in \mathbb{C}$, be the equation of a circle with center at C. If the area of the triangle, whose vertices are at the points (0, 0), C and (α, 0) is 11 square units, then α^2 equals

(1) 100 (2) 50

(3) $\frac{121}{25}$ (4) $\frac{81}{25}$

Ans. **(1)**

Sol. $\left|\frac{\bar{z}-i}{2\bar{z}+i}\right|=\frac{1}{3}$

$\left|\frac{\bar{z}-i}{\bar{z}+\frac{i}{2}}\right|=\frac{2}{3}$

$3|x-iy-i| = 2\,|x-iy+\frac{i}{2}|$

$9(x^2+(y+1)^2) = 4(x^2+(y-1/3)^2)$

$9x^2+9y^2+18y+9 = 4x^2+4y^2-4y+1$

$5x^2+5y^2+22y+8=0$

$x^2+y^2+\frac{22}{5}y+\frac{8}{5}=0$

centre $\Rightarrow (0, -\frac{11}{5})$

$\left|\frac{1}{2}\begin{vmatrix}0 & 0 & 1\\ 0 & -11/5 & 1\\ \alpha & 0 & 1\end{vmatrix}\right|=11$

$\Rightarrow \left(-\frac{11}{5}\alpha\right)^2=(11\times 2)^2$

$\Rightarrow \alpha^2 = 100$

11. Let R = {(1, 2), (2, 3), (3,3)} be a relation defined on the set {1, 2, 3, 4}. Then the minimum number of elements, needed to be added in R so the R becomes an equivalence relation, is :

(1) 10 (2) 8

(3) 9 (4) 7

Ans. **(4)**

Sol. A = {1, 2, 3, 4}

For relation to be reflexive

R = {(1,2), (2, 3), (3,3)}

Minimum elements added will be

(1,1), (2,2), (4,4) (2,1) (3,2) (3,2) (3,1) (1,3)

∴ Minimum number of elements = 7

Option : (4)

12. The number of words, which can be formed using all the letters of the word "DAUGHTER", so that all the vowels never come together, is

(1) 34000 (2) 37000

(3) 36000 (4) 35000

Ans. **(3)**

Sol. DAUGHTER

Total words = 8!

Total words in which vowels are together = 6! × 3!

words in which all vowels are not together

= 8! – 6! × 3!

= 6! [56 – 6]

= 720 × 50

= 36000

Ans.(3)

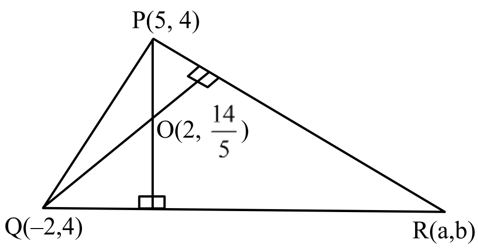
13. Let the area of a ΔPQR with vertices P(5, 4), Q(–2, 4) and R(a, b) be 35 square units. If its orthocenter and centroid are $O\left(2, \frac{14}{5}\right)$ and C(c, d) respectively, then c + 2d is equal to

(1) $\frac{7}{3}$ (2) 3

(3) 2 (4) $\frac{8}{3}$

Ans. **(2)**

Sol.



Equation of lines QR = 5x + 2y + 2 = 0

Equation of lines PR = 10x –3y – 38 = 0

∴ Point R (2, –6)

Centroid = $\left(\frac{5-2+2}{3}, \frac{4+4-6}{3}\right)$

$= \left(\frac{5}{3}, \frac{2}{3}\right)$

$c + 2d = \frac{5}{3}+\frac{4}{3}=3$

14. If $\frac{\pi}{2} \le x \le \frac{3\pi}{4}$, then $\cos^{-1}\left(\frac{12}{13}\cos x + \frac{5}{13}\sin x\right)$ is equal to

(1) $x - \tan^{-1}\frac{4}{3}$ (2) $x - \tan^{-1}\frac{5}{12}$

(3) $x + \tan^{-1}\frac{4}{5}$ (4) $x + \tan^{-1}\frac{5}{12}$

Ans. (2)

Sol. $\frac{\pi}{2} \le x \le \frac{3\pi}{4}$

$\cos^{-1}\left(\frac{12}{13}\cos x + \frac{5}{12}\sin x\right)$

$\cos^{-1}(\cos x \cos\alpha + \sin x \sin\alpha)$

$\cos^{-1}(\cos(x-\alpha))$

$\Rightarrow x - \alpha$ because $x - \alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\Rightarrow x - \tan^{-1}\frac{5}{12}$.

15. The value of $(\sin 70°)(\cot 10° \cot 70° - 1)$ is

(1) 1 (2) 0

(3) 3/2 (4) 2/3

Ans. (1)

Sol. $\sin 70°(\cot 10° \cot 70° - 1)$

$\Rightarrow \frac{\cos(80°)}{\sin 10} = 1$

16. Marks obtains by all the students of class 12 are presented in a frequency distribution with classes of equal width. Let the median of this grouped data be 14 with median class interval 12-18 and median class frequency 12. If the number of students whose marks are less than 12 is 18, then the total number of students is

(1) 48 (2) 44

(3) 40 (4) 52

Ans. (2)

Sol. medain $= \ell + \left(\frac{\frac{N}{2} - F}{f}\right) \times h$

$= 12 + \left(\frac{\frac{N}{2} - 18}{12}\right) \times 6 = 14$

$\Rightarrow \left(\frac{\frac{N}{2} - 18}{12}\right) \times 6 = 2$

$\frac{N}{2} - 18 = 4 \Rightarrow N = 44$

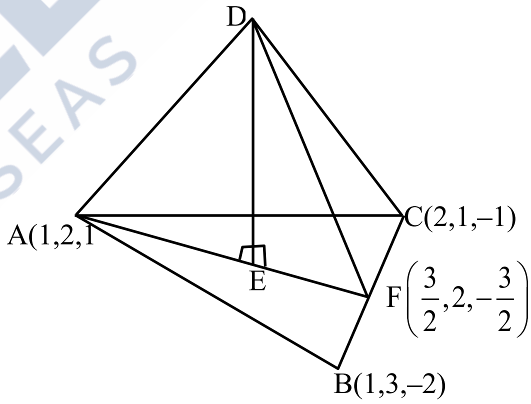
17. Let the position vectors of the vertices A, B and C of a tetrahedron ABCD be $\hat{i} + 2\hat{j} + \hat{k}$, $\hat{i} + 3\hat{j} - 2\hat{k}$ and $2\hat{i} + \hat{j} - \hat{k}$ respectively. The altitude from the vertex D to the opposite face ABC meets the median line segment through A of the triangle ABC at the point E. If the length of AD is $\frac{\sqrt{110}}{3}$ and the volume of the tetrahedron is $\frac{\sqrt{805}}{6\sqrt{2}}$, then the position vector of E is

(1) $\frac{1}{2}(\hat{i} + 4\hat{j} + 7\hat{k})$ (2) $\frac{1}{12}(7\hat{i} + 4\hat{j} + 3\hat{k})$

(3) $\frac{1}{6}(12\hat{i} + 12\hat{j} + \hat{k})$ (4) $\frac{1}{6}(7\hat{i} + 12\hat{j} + \hat{k})$

Ans. (4)

Sol.



Area of $\Delta ABC = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$

$= \frac{1}{2}\left|5\hat{i} + 3\hat{j} + \hat{k}\right| = \frac{1}{2}\sqrt{35}$

volume of tetrahedron

$= \frac{1}{3} \times \text{Base area} \times h = \frac{\sqrt{805}}{6\sqrt{2}}$

$\frac{1}{3} \times \frac{1}{2}\sqrt{35} \times h = \frac{\sqrt{805}}{6\sqrt{2}}$

$h = \sqrt{\frac{23}{2}}$

$AE^2 = AD^2 - DE^2 = \frac{13}{18} \therefore AE = \sqrt{\frac{13}{18}}$

$$\overrightarrow{AE} = |AE|.\left(\frac{\hat{i}-5\hat{k}}{\sqrt{26}}\right)$$

$$= \sqrt{\frac{13}{18}}.\left(\frac{\hat{i}-5\hat{k}}{\sqrt{26}}\right)$$

$$= \sqrt{\frac{13}{18}}.\left(\frac{\hat{i}-5\hat{k}}{\sqrt{26}}\right) = \frac{\hat{i}-5\hat{k}}{6}$$

P.V. of E $= \frac{\hat{i}-5\hat{k}}{6} + \hat{i}+2\hat{j}+\hat{k} = \frac{1}{6}\left(7\hat{i}+12\hat{j}+\hat{k}\right)$

18. If A, B and $(\text{adj}(A^{-1}) + \text{adj}(B^{-1}))$ are non-singular matrices of same order, then the inverse of $A(\text{adj}(A^{-1}) + \text{adj}(B^{-1}))^{-1}B$, is equal to

(1) $AB^{-1} + A^{-1}B$ (2) $\text{adj}(B^{-1}) + \text{adj}(A^{-1})$

(3) $\frac{1}{|AB|}(\text{adj}(B) + \text{adj}(A))$ (4) $\frac{AB^{-1}}{|A|} + \frac{BA^{-1}}{|B|}$

Ans. (3)

Sol. $\left[A\left(\text{adj}\left(A^{-1}\right)+\text{adj}\left(B^{-1}\right)\right)^{-1}.B\right]^{-1}$

$B^{-1}.\left(\text{adj}\left(A^{-1}\right)+\text{adj}\left(B^{-1}\right)\right).A^{-1}$

$B^{-1}\text{adj}\left(A^{-1}\right)A^{-1}+B^{-1}\left(\text{adj}\left(B^{-1}\right)\right).A^{-1}$

$B^{-1}\left|A^{-1}\right|I+\left|B^{-1}\right|IA^{-1}$

$\frac{B^{-1}}{|A|}+\frac{A^{-1}}{|B|}$

$\Rightarrow \frac{\text{adj}B}{|B||A|}+\frac{\text{adj}A}{|A||B|}$

$= \frac{1}{|A||B|}(\text{adj}B+\text{adj}A)$

19. If the system of equations

$(\lambda - 1)x + (\lambda - 4)y + \lambda z = 5$

$\lambda x + (\lambda - 1)y + (\lambda - 4)z = 7$

$(\lambda + 1)x + (\lambda + 2)y - (\lambda + 2)z = 9$

has infinitely many solutions, then $\lambda^2 + \lambda$ is equal to

(1) 10 (2) 12

(3) 6 (4) 20

Ans. (2)

Sol. $(\lambda-1)x+(\lambda-4)y+\lambda z=5$

$\lambda x+(\lambda-1)y+(\lambda-4)z=7$

$(\lambda+1)x+(\lambda+2)y-(\lambda+2)z=9$

For infinitely many solutions

$$D = \begin{vmatrix} \lambda-1 & \lambda-4 & \lambda \\ \lambda & \lambda-1 & \lambda-4 \\ \lambda+1 & \lambda+2 & -(\lambda+2) \end{vmatrix} = 0$$

$(\lambda - 3)(2\lambda + 1) = 0$

$$D_x = \begin{vmatrix} 5 & \lambda-4 & \lambda \\ 7 & \lambda-1 & \lambda-4 \\ 9 & \lambda+2 & -(\lambda+2) \end{vmatrix} = 0$$

$2(3 - \lambda)(23 - 2\lambda) = 0$

$\lambda = 3$

$\therefore \lambda^2 + \lambda = 9 + 3 = 12$

20. One die has two faces marked 1, two faces marked 2, one face marked 3 and one face marked 4. Another die has one face marked 1, two faces marked 2, two faces marked 3 and one face marked 4. The probability of getting the sum of numbers to be 4 or 5, when both the dice are thrown together, is

(1) $\frac{1}{2}$ (2) $\frac{3}{5}$

(3) $\frac{2}{3}$ (4) $\frac{4}{9}$

Ans. (1)

Sol. a = number or dice 1

b = number on dice 2

(a,b) = (1,3), (3,1), (2,2), (2,3), (3,2), (1,4), (4,1)

Required probability

$$= \frac{2}{6}\times\frac{2}{6}+\frac{1}{6}\times\frac{1}{6}+\frac{2}{6}\times\frac{2}{6}+\frac{2}{6}\times\frac{2}{6}+\frac{1}{6}\times\frac{2}{6}+\frac{2}{6}\times\frac{1}{6}+\frac{1}{6}\times\frac{2}{6}$$

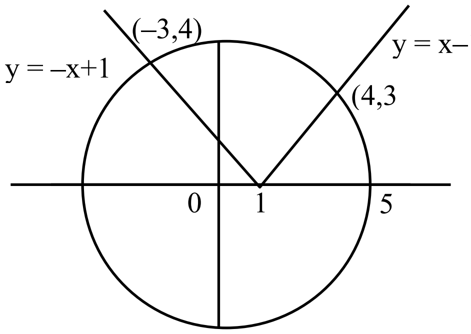
$$= \frac{18}{36} = \frac{1}{2}$$

SECTION-B

21. If the area of the larger portion bounded between the curves $x^2 + y^2 = 25$ and $y = |x - 1|$ is $\frac{1}{4}(b\pi + c)$, $b, c \in \mathbb{N}$, then $b + c$ is equal to ________

Ans. (77)

Sol.



$x^2 + y^2 = 5$

$x^2 + (x - 1)^2 = 25 \Rightarrow x = 4$

$x^2 + (-x + 1)^2 = 5 \Rightarrow x = -3$

$$A = 25\pi - \int_{-3}^{4}\sqrt{25 - x^2}\,dx + \frac{1}{2}\times 4\times 4 + \frac{1}{2}\times 3\times 3$$

$$A = 25\pi + \frac{25}{2} - \left[\frac{x}{2}\sqrt{25 - x^2} + \frac{25}{2}\sin^{-1}\frac{x}{5}\right]_{-3}^{4}$$

$$A = 25\pi + \frac{25}{2} - \left[6 + \frac{25}{2}\sin^{-1}\frac{4}{5} + 6 + \frac{25}{2}\sin^{-1}\frac{3}{5}\right]$$

$$A = 25\pi + \frac{1}{2} - \frac{25}{2}\cdot\frac{\pi}{2}$$

$$A = \frac{75\pi}{4} + \frac{1}{2}$$

$$A = \frac{1}{4}(75\pi + 2)$$

$b = 75$, $c = 2$

$b + c = 75 + 2 = 77$

22. The sum of all rational terms in the expansion of $(1 + 2^{1/3} + 3^{1/2})^6$ is equal to _______

Ans. (612)

Sol. $\left(1 + 2^{\frac{1}{3}} + 3^{\frac{1}{2}}\right)^6$

$$= \frac{\underline{|6}}{\underline{|r_1}\,\underline{|r_2}\,\underline{|r_3}}(1)^{r_1}(2)^{\frac{r_2}{3}}(3)^{\frac{r_3}{2}}$$

r_1	r_2	r_3
6	0	0
4	0	2
2	0	4
0	0	6
3	3	0
1	3	2
0	6	0

$$\text{sum} = \frac{\underline{|6}}{\underline{|6}\,\underline{|0}\,\underline{|0}} + \frac{\underline{|6}}{\underline{|4}\,\underline{|0}\,\underline{|2}}(3) + \frac{\underline{|6}}{\underline{|2}\,\underline{|0}\,\underline{|4}}(3)^2 + \frac{\underline{|6}}{\underline{|0}\,\underline{|0}\,\underline{|6}}(3)^3$$

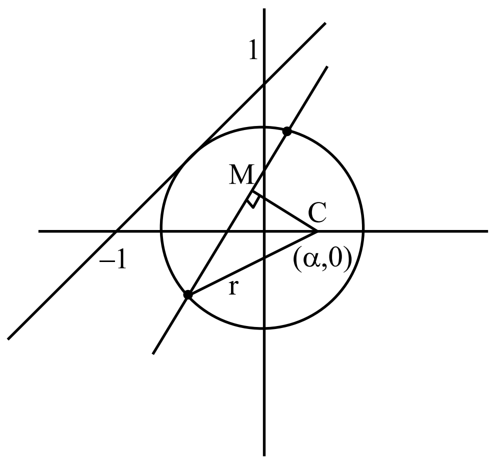
$$+ \frac{\underline{|6}}{\underline{|3}\,\underline{|3}\,\underline{|0}}(2) + \frac{\underline{|6}}{\underline{|1}\,\underline{|3}\,\underline{|2}}(2)^1(3)^1 + \frac{\underline{|6}}{\underline{|0}\,\underline{|6}\,\underline{|0}}(2)^2$$

$= 1 + 45 + 135 + 27 + 40 + 360 + 4 = 612$

23. Let the circle C touch the line $x - y + 1 = 0$, have the centre on the positive x-axis, and cut off a chord of length $\frac{4}{\sqrt{13}}$ along the line $-3x + 2y = 1$. Let H be the hyperbola $\frac{x^2}{\alpha^2} - \frac{y^2}{\beta^2} = 1$, whose one of the foci is the centre of C and the length of the transverse axis is the diameter of C. Then $2\alpha^2 + 3\beta^2$ is equal to ________

Ans. (19)

Sol.



$x - y + 1 = 0$

$p = r$

$$\left|\frac{\alpha - 0 + 1}{\sqrt{2}}\right| = r \Rightarrow (\alpha + 1)^2 = 2r^2 \quad .... (1)$$

$$\text{now}\left(\frac{-3\alpha + 0 - 1}{\sqrt{9 + 4}}\right)^2 + \left(\frac{2}{\sqrt{13}}\right)^2 = r^2$$

$\Rightarrow (3\alpha + 1)^2 + 4 = 13\, r^2$ ......(2)

(1) & (2) $\Rightarrow (3\alpha + 1)^2 + 4 = 13\,\dfrac{(\alpha+1)^2}{2}$

$\Rightarrow 18\alpha^2 + 12\alpha + 2 + 8 = 13\alpha^2 + 26\alpha + 13$

$\Rightarrow 5\alpha^2 - 14\alpha - 3 = 0$

$\Rightarrow 5\alpha^2 - 15\alpha + \alpha - 3 = 0$

$\Rightarrow 5\alpha^2 - 15\alpha + \alpha - 3 = 0$

$\Rightarrow \alpha = \dfrac{-1}{5}, 3$

$\therefore \quad r = 2\sqrt{2}$

How $\alpha e = 3$ and $2\alpha = 4\sqrt{2}$

$\alpha^2 e^2 = 9 \Rightarrow \alpha = 2\sqrt{2} \Rightarrow \alpha^2 = 8$

$\alpha^2\left(1 + \dfrac{\beta^2}{\alpha^2}\right) = 9$

$\alpha^2 + \beta^2 = 9$

$\therefore \beta^2 = 1$

$\therefore 2\alpha^2 + 3\beta^2 = 2\,(8) + 3(1) = 19$

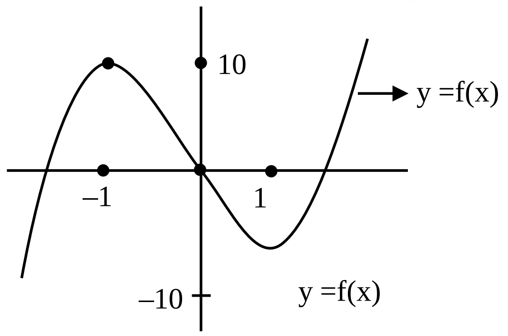
24. If the set of all values of a, for which the equation $5x^3 - 15x - a = 0$ has three distinct real roots, is the interval (α, β), then $\beta - 2\alpha$ is equal to ________

Ans. (30)

Sol. $5x^3 - 15x - a = 0$

$f(x) = 5x^3 - 15x$

$f'(x) = 15x^2 - 15 = 15(x-1)(x+1)$



$a \in (-10, 10)$

$\alpha = -10,\ \beta = 10$

$\beta - 2\alpha = 10 + 20 = 30$

25. If the equation $a(b - c)x^2 + b(c - a)x + c(a - b) = 0$ has equal roots, where $a + c = 15$ and $b = \dfrac{36}{5}$, then $a^2 + c^2$ is equal to ________

Ans. (117)

Sol. $a(b - c)\, x^2 + b\,(c - a)\, x + c(a - b) = 0$

$x = 1$ is root $\therefore$ other root is 1

$\alpha + \beta = -\dfrac{b(c-a)}{a(b-c)} = 2$

$\Rightarrow -bc + ab = 2ab - 2ac$

$\Rightarrow 2ac = ab + bc$

$\Rightarrow 2ac = b(a + c)$

$\Rightarrow 2ac = 15b$ ... (1)

$\Rightarrow 2ac = 15\left(\dfrac{36}{5}\right) = 108$

$\Rightarrow ac = 54$

$a + c = 15$

$a^2 + c^2 + 2ac = 225$

$a^2 + c^2 = 225 - 108 = 117$