FINAL JEE–MAIN EXAMINATION – JANUARY, 2024

(Held On Monday 29th January, 2024) **TIME : 3 : 00 PM to 6 : 00 PM**

MATHEMATICS **TEST PAPER WITH SOLUTION**

SECTION-A

1. Let $A = \begin{bmatrix} 2 & 1 & 2 \\ 6 & 2 & 11 \\ 3 & 3 & 2 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 2 & 0 \\ 5 & 0 & 2 \\ 7 & 1 & 5 \end{bmatrix}$. The sum of the prime factors of $|P^{-1}AP - 2I|$ is equal to

(1) 26 (2) 27 (3) 66 (4) 23

Ans. **(1)**

Sol. $|P^{-1}AP - 2I| = |P^{-1}AP - 2P^{-1}P|$

$= |P^{-1}(A - 2I)P|$

$= |P^{-1}||A - 2I||P|$

$= |A-2I|$

$= \begin{vmatrix} 0 & 1 & 2 \\ 6 & 0 & 11 \\ 3 & 3 & 0 \end{vmatrix} = 69$

So, Prime factor of 69 is 3 & 23

So, sum = 26

2. Number of ways of arranging 8 identical books into 4 identical shelves where any number of shelves may remain empty is equal to

(1) 18 (2) 16 (3) 12 (4) 15

Ans. **(4)**

Sol. 3 Shelf empty : $(8, 0, 0, 0) \to 1\text{way}$

2 shelf empty : $\left.\begin{matrix}(7,1,0,0)\\(6,2,0,0)\\(5,3,0,0)\\(4,4,0,0)\end{matrix}\right] \to 4\text{ways}$

1 shelf empty : $\left.\begin{matrix}(6,1,1,0) & (3,3,2,0)\\(5,2,1,0) & (4,2,2,0)\\(4,3,1,0) & \end{matrix}\right] \to 5\text{ways}$

0 Shelf empty : $\left.\begin{matrix}(1,2,3,2) & (5,1,1,1)\\(2,2,2,2) & \\(3,3,1,1) & \\(4,2,1,1) & \end{matrix}\right] \to 5\text{ways}$

Total = 15 ways

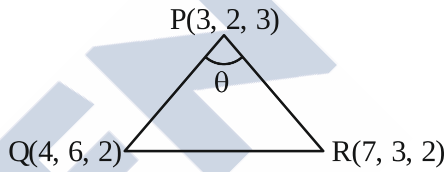
3. Let P(3, 2, 3), Q (4, 6, 2) and R (7, 3, 2) be the vertices of ΔPQR. Then, the angle $\angle QPR$ is

(1) $\frac{\pi}{6}$ (2) $\cos^{-1}\left(\frac{7}{18}\right)$

(3) $\cos^{-1}\left(\frac{1}{18}\right)$ (4) $\frac{\pi}{3}$

Ans. **(4)**

Sol.



Direction ratio of PR = (4, 1, -1)

Direction ratio of PQ = (1, 4, -1)

Now, $\cos\theta = \left|\frac{4+4+1}{\sqrt{18}\cdot\sqrt{18}}\right|$

$\theta = \frac{\pi}{3}$

4. If the mean and variance of five observations are $\frac{24}{5}$ and $\frac{194}{25}$ respectively and the mean of first four observations is $\frac{7}{2}$, then the variance of the first four observations in equal to

(1) $\frac{4}{5}$ (2) $\frac{77}{12}$ (3) $\frac{5}{4}$ (4) $\frac{105}{4}$

Ans. **(3)**

Sol. $\bar{X} = \frac{24}{5}; \sigma^2 = \frac{194}{25}$

Let first four observation be x_1, x_2, x_3, x_4

Here, $\frac{x_1 + x_2 + x_3 + x_4 + x_5}{5} = \frac{24}{5}$ ......(1)

Also, $\frac{x_1 + x_2 + x_3 + x_4}{4} = \frac{7}{2}$

$\Rightarrow \boxed{x_1 + x_2 + x_3 + x_4 = 14}$

Now from eqn -1

$x_5 = 10$

Now, $\sigma^2 = \frac{194}{25}$

$$\frac{x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2}{5} - \frac{576}{25} = \frac{194}{25}$$

$$\Rightarrow x_1^2 + x_2^2 + x_3^2 + x_4^2 = 54$$

Now, variance of first 4 observations

$$\text{Var} = \frac{\sum_{i=1}^{4} x_i^2}{4} - \left(\frac{\sum_{i=1}^{4} x_i}{4}\right)^2$$

$$= \frac{54}{4} - \frac{49}{4} = \frac{5}{4}$$

5. The function $f(x) = 2x + 3(x)^{\frac{2}{3}}, x \in \mathbb{R}$, has

(1) exactly one point of local minima and no point of local maxima

(2) exactly one point of local maxima and no point of local minima

(3) exactly one point of local maxima and exactly one point of local minima

(4) exactly two points of local maxima and exactly one point of local minima

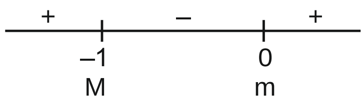
Ans. (3)

Sol. $f(x) = 2x + 3(x)^{\frac{2}{3}}$

$$f'(x) = 2 + 2x^{\frac{-1}{3}}$$

$$= 2\left(1 + \frac{1}{x^{\frac{1}{3}}}\right)$$

$$= 2\left(\frac{x^{\frac{1}{3}} + 1}{x^{\frac{1}{3}}}\right)$$



So, maxima (M) at x = -1 & minima(m) at x = 0

6. Let r and θ respectively be the modulus and amplitude of the complex number $z = 2 - i\left(2\tan\frac{5\pi}{8}\right)$, then (r, θ) is equal to

(1) $\left(2\sec\frac{3\pi}{8}, \frac{3\pi}{8}\right)$ (2) $\left(2\sec\frac{3\pi}{8}, \frac{5\pi}{8}\right)$

(3) $\left(2\sec\frac{5\pi}{8}, \frac{3\pi}{8}\right)$ (4) $\left(2\sec\frac{11\pi}{8}, \frac{11\pi}{8}\right)$

Ans. (1)

Sol. $z = 2 - i\left(2\tan\frac{5\pi}{8}\right) = x + iy$ (let)

$$r = \sqrt{x^2 + y^2} \ \& \ \theta = \tan^{-1}\frac{y}{x}$$

$$r = \sqrt{(2)^2 + \left(2\tan\frac{5\pi}{8}\right)^2}$$

$$= \left|2\sec\frac{5\pi}{8}\right| = \left|2\sec\left(\pi - \frac{3\pi}{8}\right)\right|$$

$$= 2\sec\frac{3\pi}{8}$$

$$\& \quad \theta = \tan^{-1}\left(\frac{-2\tan\frac{5\pi}{8}}{2}\right)$$

$$= \tan^{-1}\left(\tan\left(\pi - \frac{5\pi}{8}\right)\right)$$

$$= \frac{3\pi}{8}$$

7. The sum of the solutions $x \in \mathbb{R}$ of the equation $\frac{3\cos 2x + \cos^3 2x}{\cos^6 x - \sin^6 x} = x^3 - x^2 + 6$ is

(1) 0 (2) 1

(3) –1 (4) 3

Ans. (3)

Sol. $\frac{3\cos 2x + \cos^3 2x}{\cos^6 x - \sin^6 x} = x^3 - x^2 + 6$

$$\Rightarrow \frac{\cos 2x\,(3 + \cos^2 2x)}{\cos 2x\,(1 - \sin^2 x\cos^2 x)} = x^3 - x^2 + 6$$

$$\Rightarrow \frac{4(3 + \cos^2 2x)}{(4 - \sin^2 2x)} = x^3 - x^2 + 6$$

$$\Rightarrow \frac{4(3 + \cos^2 2x)}{(3 + \cos^2 2x)} = x^3 - x^2 + 6$$

$x^3 - x^2 + 2 = 0 \Rightarrow (x + 1)(x^2 - 2x + 2) = 0$

so, sum of real solutions = –1

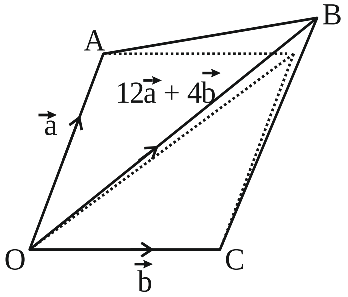
8. Let $\overrightarrow{OA} = \vec{a}, \overrightarrow{OB} = 12\vec{a} + 4\vec{b}$ and $\overrightarrow{OC} = \vec{b}$, where O is the origin. If S is the parallelogram with adjacent sides OA and OC, then

$\dfrac{\text{area of the quadrilateral OABC}}{\text{area of S}}$ is equal to ___

(1) 6 (2) 10
(3) 7 (4) 8

Ans. **(4)**

Sol.



Area of parallelogram, $S = |\vec{a} \times \vec{b}|$

Area of quadrilateral =Area(ΔOAB)+Area (ΔOBC)

$= \frac{1}{2}\left\{|\vec{a} \times (12\vec{a} + 4\vec{b})| + |\vec{b} \times (12\vec{a} + 4\vec{b})|\right\}$

$= 8|(\vec{a} \times \vec{b})|$

$\text{Ratio} = \dfrac{8|(\vec{a} \times \vec{b})|}{|(\vec{a} \times \vec{b})|} = 8$

9. If $\log_e a$, $\log_e b$, $\log_e c$ are in an A.P. and $\log_e a - \log_e 2b$, $\log_e 2b - \log_e 3c$, $\log_e 3c - \log_e a$ are also in an A.P, then a : b : c is equal to

(1) 9 : 6 : 4 (2) 16 : 4 : 1
(3) 25 : 10 : 4 (4) 6 : 3 : 2

Ans. **(1)**

Sol. $\log_e a, \log_e b, \log_e c$ are in A.P.

$\therefore b^2 = ac$ .....(i)

Also

$\log_e\left(\frac{a}{2b}\right), \log_e\left(\frac{2b}{3c}\right), \log_e\left(\frac{3c}{a}\right)$ are in A.P.

$\left(\frac{2b}{3c}\right)^2 = \frac{a}{2b} \times \frac{3c}{a}$

$\frac{b}{c} = \frac{3}{2}$

Putting in eq. (i) $b^2 = a \times \frac{2b}{3}$

$\frac{a}{b} = \frac{3}{2}$

$a : b : c = 9 : 6 : 4$

10. If

$$\int \frac{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x}{\sqrt{\sin^3 x \cos^3 x \sin(x-\theta)}} dx = A\sqrt{\cos\theta \tan x - \sin\theta} + B\sqrt{\cos\theta - \sin\theta \cot x} + C,$$

where C is the integration constant, then AB is equal to

(1) $4\operatorname{cosec}(2\theta)$ (2) $4\sec\theta$
(3) $2\sec\theta$ (4) $8\operatorname{cosec}(2\theta)$

Ans. **(4)**

Sol. $\displaystyle\int \frac{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x}{\sqrt{\sin^3 x \cos^3 x \sin(x-\theta)}} dx$

$$I = \int \frac{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x}{\sqrt{\sin^3 x \cos^3 x\, (\sin x \cos\theta - \cos x \sin\theta)}} dx$$

$$= \int \frac{\sin^{\frac{3}{2}} x}{\sin^{\frac{3}{2}} x \cos^2 x \sqrt{\tan x \cos\theta - \sin\theta}} dx + \int \frac{\cos^{\frac{3}{2}} x}{\sin^2 x \cos^{\frac{3}{2}} x \sqrt{\cos\theta - \cot x \sin\theta}} dx =$$

$$\int \frac{\sec^2 x}{\sqrt{\tan x \cos\theta - \sin\theta}} dx + \int \frac{\operatorname{cosec}^2 x}{\sqrt{\cos\theta - \cot x \sin\theta}} dx$$

$I = I_1 + I_2$ ...... {Let}

For I_1, let $\tan x \cos\theta - \sin\theta = t^2$

$\sec^2 x\, dx = \frac{2t\, dt}{\cos\theta}$

For I_2, let $\cos\theta - \cot x \sin\theta = z^2$

$\operatorname{cosec}^2 x\, dx = \frac{2z\, dz}{\sin\theta}$

$I = I_1 + I_2$

$= \int \frac{2t\, dt}{\cos\theta\ t} + \int \frac{2z\, dz}{\sin\theta\ z}$

$= \frac{2t}{\cos\theta} + \frac{2z}{\sin\theta}$

$= 2\sec\theta\sqrt{\tan x \cos\theta - \sin\theta} + 2\operatorname{cosec}\theta\sqrt{\cos\theta - \cot x \sin\theta}$

Comparing

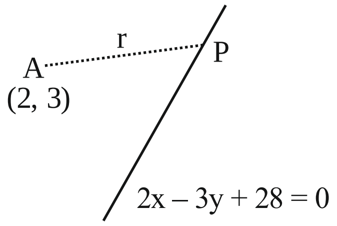
$AB = 8\operatorname{cosec}2\theta$

11. The distance of the point (2, 3) from the line $2x - 3y + 28 = 0$, measured parallel to the line $\sqrt{3}x - y + 1 = 0$, is equal to

(1) $4\sqrt{2}$ (2) $6\sqrt{3}$
(3) $3 + 4\sqrt{2}$ (4) $4 + 6\sqrt{3}$

Ans. (4)

Sol.



Writing P in terms of parametric co-ordinates 2 + r cos θ, 3 + r sin θ as tan θ = $\sqrt{3}$

$$P(2+\frac{r}{2},3+\frac{\sqrt{3}r}{2})$$

P must satisfy 2x – 3y + 28 = 0

$$\text{So, } 2(2+\frac{r}{2})-3(3+\frac{\sqrt{3}r}{2})+28=0$$

We find r = $4+6\sqrt{3}$

12. If $\sin\left(\frac{y}{x}\right)=\log_e|x|+\frac{\alpha}{2}$ is the solution of the differential equation $x\cos\left(\frac{y}{x}\right)\frac{dy}{dx}=y\cos\left(\frac{y}{x}\right)+x$ and $y(1)=\frac{\pi}{3}$, then α^2 is equal to

(1) 3 (2) 12

(3) 4 (4) 9

Ans. (1)

Sol. Differential equation :–

$$x\cos\frac{y}{x}\frac{dy}{dx}=y\cos\frac{y}{x}+x$$

$$\cos\frac{y}{x}\left[x\frac{dy}{dx}-y\right]=x$$

Divide both sides by x^2

$$\cos\frac{y}{x}\left(\frac{x\frac{dy}{dx}-y}{x^2}\right)=\frac{1}{x}$$

Let $\frac{y}{x}=t$

$$\cos t\left(\frac{dt}{dx}\right)=\frac{1}{x}$$

$$\cos t\, dt=\frac{1}{x}dx$$

Integrating both sides

$$\sin t=\ln|x|+c$$

$$\sin\frac{y}{x}=\ln|x|+c$$

Using $y(1)=\frac{\pi}{3}$, we get $c=\frac{\sqrt{3}}{2}$

So, $\alpha=\sqrt{3}\Rightarrow\alpha^2=3$

13. If each term of a geometric progression $a_1, a_2, a_3,\ldots$ with $a_1=\frac{1}{8}$ and $a_2\neq a_1$, is the arithmetic mean of the next two terms and $S_n = a_1 + a_2 + \ldots + a_n$, then $S_{20} - S_{18}$ is equal to

(1) 2^{15} (2) -2^{18}

(3) 2^{18} (4) -2^{15}

Ans. (4)

Sol. Let r'th term of the GP be ar^{n-1}. Given,

$$2a_r = a_{r+1} + a_{r+2}$$

$$2ar^{n-1} = ar^n + ar^{n+1}$$

$$\frac{2}{r}=1+r$$

$$r^2+r-2=0$$

Hence, we get, r = – 2 (as r ≠ 1)

So, $S_{20} - S_{18}$ = (Sum upto 20 terms) – (Sum upto 18 terms) = $T_{19} + T_{20}$

$$T_{19} + T_{20} = ar^{18}(1 + r)$$

Putting the values a = $\frac{1}{8}$ and r = – 2;

we get $T_{19} + T_{20} = -2^{15}$

14. Let A be the point of intersection of the lines 3x + 2y = 14, 5x – y = 6 and B be the point of intersection of the lines 4x + 3y = 8, 6x + y = 5. The distance of the point P(5, –2) from the line AB is

(1) $\frac{13}{2}$ (2) 8 (3) $\frac{5}{2}$ (4) 6

Ans. (4)

Sol. Solving lines L_1 $(3x + 2y = 14)$ and L_2 $(5x - y = 6)$ to get A(2, 4) and solving lines L_3 $(4x + 3y = 8)$ and L_4 $(6x + y = 5)$ to get $B\left(\frac{1}{2},2\right)$.

Finding Eqn. of AB : $4x - 3y + 4 = 0$

Calculate distance PM

$$\Rightarrow \left|\frac{4(5)-3(-2)+4}{5}\right| = 6$$

15. Let $x = \frac{m}{n}$ (m, n are co-prime natural numbers) be a solution of the equation $\cos\left(2\sin^{-1}x\right) = \frac{1}{9}$ and let $\alpha, \beta(\alpha > \beta)$ be the roots of the equation $mx^2 - nx - m + n = 0$. Then the point (α,β) lies on the line

(1) $3x + 2y = 2$ (2) $5x - 8y = -9$

(3) $3x - 2y = -2$ (4) $5x + 8y = 9$

Ans. **(4)**

Sol. Assume $\sin^{-1}x = \theta$

$$\cos(2\theta) = \frac{1}{9}$$

$$\sin\theta = \pm\frac{2}{3}$$

as m and n are co-prime natural numbers,

$$x = \frac{2}{3}$$

i.e. m = 2, n = 3

So, the quadratic equation becomes $2x^2 - 3x + 1 = 0$ whose roots are $\alpha = 1, \beta = \frac{1}{2}$

$\left(1,\frac{1}{2}\right)$ lies on $5x + 8y = 9$

16. The function $f(x) = \frac{x}{x^2-6x-16}, x \in \mathbb{R} - \{-2, 8\}$

(1) decreases in (–2, 8) and increases in $(-\infty,-2)\cup(8,\infty)$

(2) decreases in $(-\infty,-2)\cup(-2,8)\cup(8,\infty)$

(3) decreases in $(-\infty,-2)$ and increases in $(8,\infty)$

(4) increases in $(-\infty,-2)\cup(-2,8)\cup(8,\infty)$

Ans. **(2)**

Sol. $f(x) = \frac{x}{x^2-6x-16}$

Now,

$$f'(x) = \frac{-(x^2+16)}{(x^2-6x-16)^2}$$

$f'(x) < 0$

Thus f(x) is decreasing in

$(-\infty,-2)\cup(-2,8)\cup(8,\infty)$

17. Let $y = \log_e\left(\frac{1-x^2}{1+x^2}\right)$, $-1 < x < 1$. Then at $x = \frac{1}{2}$, the value of $225(y' - y'')$ is equal to

(1) 732 (2) 746

(3) 742 (4) 736

Ans. **(4)**

Sol. $y = \log_e\left(\frac{1-x^2}{1+x^2}\right)$

$$\frac{dy}{dx} = y' = \frac{-4x}{1-x^4}$$

Again,

$$\frac{d^2y}{dx^2} = y'' = \frac{-4(1+3x^4)}{(1-x^4)^2}$$

Again

$$y' - y'' = \frac{-4x}{1-x^4} + \frac{4(1+3x^4)}{(1-x^4)^2}$$

at $x = \frac{1}{2}$,

$$y' - y'' = \frac{736}{225}$$

Thus $225(y' - y'') = 225 \times \frac{736}{225} = 736$

18. If R is the smallest equivalence relation on the set {1, 2, 3, 4} such that $\{(1,2), (1,3)\} \subset R$, then the number of elements in R is ______

(1) 10 (2) 12

(3) 8 (4) 15

Ans. **(1)**

Sol. Given set {1, 2, 3, 4}

Minimum order pairs are

(1, 1), (2, 2), (3, 3), (4, 4), (3, 1),(2, 1), (2, 3),(3,2), (1, 3), (1, 2)

Thus no. of elements = 10

19. An integer is chosen at random from the integers 1, 2, 3, …, 50. The probability that the chosen integer is a multiple of atleast one of 4, 6 and 7 is

(1) $\frac{8}{25}$ (2) $\frac{21}{50}$

(3) $\frac{9}{50}$ (4) $\frac{14}{25}$

Ans. **(2)**

Sol. Given set = {1, 2, 3, …….. 50}

P(A) = Probability that number is multiple of 4

P(B) = Probability that number is multiple of 6

P(C) = Probability that number is multiple of 7

Now,

$$P(A)=\frac{12}{50},\ P(B)=\frac{8}{50},\ P(C)=\frac{7}{50}$$

again

$$P(A\cap B)=\frac{4}{50}, P(B\cap C)=\frac{1}{50}, P(A\cap C)=\frac{1}{50}$$

$$P(A\cap B\cap C)=0$$

Thus

$$P(A\cup B\cup C)=\frac{12}{50}+\frac{8}{50}+\frac{7}{50}-\frac{4}{50}-\frac{1}{50}-\frac{1}{50}+0$$

$$=\frac{21}{50}$$

20. Let a unit vector $\hat{u}=x\hat{i}+y\hat{j}+z\hat{k}$ make angles $\frac{\pi}{2},\frac{\pi}{3}$ and $\frac{2\pi}{3}$ with the vectors $\frac{1}{\sqrt{2}}\hat{i}+\frac{1}{\sqrt{2}}\hat{k}$, $\frac{1}{\sqrt{2}}\hat{j}+\frac{1}{\sqrt{2}}\hat{k}$ and $\frac{1}{\sqrt{2}}\hat{i}+\frac{1}{\sqrt{2}}\hat{j}$ respectively. If $\vec{v}=\frac{1}{\sqrt{2}}\hat{i}+\frac{1}{\sqrt{2}}\hat{j}+\frac{1}{\sqrt{2}}\hat{k}$, then $|\hat{u}-\vec{v}|^2$ is equal to

(1) $\frac{11}{2}$ (2) $\frac{5}{2}$

(3) 9 (4) 7

Ans. **(2)**

Sol. Unit vector $\hat{u}=x\hat{i}+y\hat{j}+z\hat{k}$

$$\vec{p}_1=\frac{1}{\sqrt{2}}\hat{i}+\frac{1}{\sqrt{2}}\hat{k}, \vec{p}_2=\frac{1}{\sqrt{2}}\hat{j}+\frac{1}{\sqrt{2}}\hat{k}$$

$$\vec{p}_3=\frac{1}{\sqrt{2}}\hat{i}+\frac{1}{\sqrt{2}}\hat{j}$$

Now angle between $\hat{u}$ and $\vec{p}_1=\frac{\pi}{2}$

$$\hat{u}\cdot\vec{p}_1=0\Rightarrow\frac{x}{\sqrt{2}}+\frac{z}{\sqrt{2}}=0$$

$$\Rightarrow x+z=0 \quad ....(i)$$

Angle between $\hat{u}$ and $\vec{p}_2=\frac{\pi}{3}$

$$\hat{u}\cdot\vec{p}_2=|\hat{u}|\cdot|\vec{p}_2|\cos\frac{\pi}{3}$$

$$\Rightarrow\frac{y}{\sqrt{2}}+\frac{z}{\sqrt{2}}=\frac{1}{2}\Rightarrow y+z=\frac{1}{\sqrt{2}} \quad ....(ii)$$

Angle between $\hat{u}$ and $\vec{p}_3=\frac{2\pi}{3}$

$$\hat{u}\cdot\vec{p}_3=|\hat{u}|\cdot|\vec{p}_3|\cos\frac{2\pi}{3}$$

$$\Rightarrow\frac{x}{\sqrt{2}}+\frac{4}{\sqrt{2}}=\frac{-1}{2}\Rightarrow x+y=\frac{-1}{\sqrt{2}} \quad .... (iii)$$

from equation (i), (ii) and (iii) we get

$$x=\frac{-1}{\sqrt{2}} \quad y=0 \quad z=\frac{1}{\sqrt{2}}$$

Thus $\hat{u}-\vec{v}=\frac{-1}{\sqrt{2}}\hat{i}+\frac{1}{\sqrt{2}}\hat{k}-\frac{1}{\sqrt{2}}\hat{i}-\frac{1}{\sqrt{2}}\hat{j}-\frac{1}{\sqrt{2}}\hat{k}$

$$\hat{u}-\vec{v}=\frac{-2}{\sqrt{2}}\hat{i}-\frac{1}{\sqrt{2}}\hat{j}$$

$$\therefore|\hat{u}-\vec{v}|^2=\left(\sqrt{\frac{4}{2}+\frac{1}{2}}\right)^2=\frac{5}{2}$$

SECTION-B

21. Let α, β be the roots of the equation $x^2-\sqrt{6}x+3=0$ such that Im (α) > Im (β). Let a, b be integers not divisible by 3 and n be a natural number such that $\frac{\alpha^{99}}{\beta}+\alpha^{98}=3^n(a+ib), i=\sqrt{-1}$. Then n + a + b is equal to _______.

Ans. **49**

Sol. $x^2-\sqrt{6}x+6=0 \begin{smallmatrix}\nearrow \alpha \\ \searrow \beta\end{smallmatrix}$

$x=\frac{\sqrt{6}\pm i\sqrt{6}}{2}=\frac{\sqrt{6}}{2}(1\pm i)$

$\alpha=\sqrt{3}(e^{i\frac{\pi}{4}}),\ \beta=\sqrt{3}(e^{-i\frac{\pi}{4}})$

$\therefore \frac{\alpha^{99}}{\beta}+\alpha^{98}=\alpha^{98}\left(\frac{\alpha}{\beta}+1\right)$

$=\frac{\alpha^{98}(\alpha+\beta)}{\beta}=3^{49}\left(e^{i99\frac{\pi}{4}}\right)\times\sqrt{2}$

$=3^{49}(-1+i)$

$=3^n(a+ib)$

$\therefore n=49, a=-1, b=1$

$\therefore n+a+b=49-1+1=49$

22. Let for any three distinct consecutive terms a, b, c of an A.P, the lines ax + by + c = 0 be concurrent at the point P and Q (α,β) be a point such that the system of equations

$x+y+z=6$,

$2x+5y+\alpha z=\beta$ and

$x+2y+3z=4$, has infinitely many solutions. Then $(PQ)^2$ is equal to ________.

Ans. **113**

Sol. $\because$ a, b, c and in A.P

$\Rightarrow 2b=a+c\Rightarrow a-2b+c=0$

$\therefore$ ax + by + c passes through fixed point (1, -2)

$\therefore P=(1,-2)$

For infinite solution,

$D=D_1=D_2=D_3=0$

$D:\begin{vmatrix}1&1&1\\2&5&\alpha\\1&2&3\end{vmatrix}=0$

$\Rightarrow \alpha=8$

$D_1:\begin{vmatrix}6&1&1\\\beta&5&\alpha\\4&2&3\end{vmatrix}=0\Rightarrow\beta=6$

$\therefore Q=(8,6)$

$\therefore PQ^2=113$

23. Let $P(\alpha,\beta)$ be a point on the parabola $y^2=4x$. If P also lies on the chord of the parabola $x^2=8y$ whose mid point is $\left(1,\frac{5}{4}\right)$. Then $(\alpha-28)(\beta-8)$ is equal to ___________.

Ans. **192**

Sol. Parabola is $x^2=8y$

Chord with mid point (x_1,y_1) is $T=S_1$

$\therefore xx_1-4(y+y_1)=x_1^2-8y_1$

$\therefore (x_1,y_1)=\left(1,\frac{5}{4}\right)$

$\Rightarrow x-4\left(y+\frac{5}{4}\right)=1-8\times\frac{5}{4}=-9$

$\therefore x-4y+4=0$ ...... (i)

(α,β) lies on (i) & also on $y^2=4x$

$\therefore \alpha-4\beta+4=0$ ..... (ii)

& $\beta^2=4\alpha$ ..... (iii)

Solving (ii) & (iii)

$\beta^2=4(4\beta-4)\Rightarrow\beta^2-16\beta+16=0$

$\therefore \beta=8\pm4\sqrt{3}$ and $\alpha=4\beta-4=28\pm16\sqrt{3}$

$\therefore(\alpha,\beta)=(28+16\sqrt{3},8+4\sqrt{3})$ & $(28-16\sqrt{3},8-4\sqrt{3})$

$\therefore(\alpha-28)(\beta-8)=(\pm16\sqrt{3})(\pm4\sqrt{3})$

$=192$

24. If $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\sqrt{1-\sin 2x}\,dx=\alpha+\beta\sqrt{2}+\gamma\sqrt{3}$, where α, β and γ are rational numbers, then $3\alpha+4\beta-\gamma$ is equal to _______.

Ans. **6**

Sol. $=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\sqrt{1-\sin 2x}\,dx$

$=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}|\sin x-\cos x|dx$

$$= \int_{\pi/6}^{\pi/4}(\cos x - \sin x)dx + \int_{\pi/4}^{\pi/3}(\sin x - \cos x)dx$$

$$= -1 + 2\sqrt{2} - \sqrt{3}$$

$$= \alpha + \beta\sqrt{2} + \gamma\sqrt{3}$$

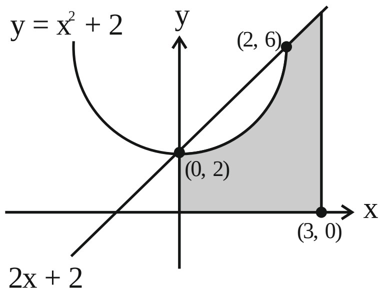
$$\alpha = -1, \beta = 2, \gamma = -1$$

$$3\alpha + 4\beta - \gamma = 6$$

25. Let the area of the region $\{(x, y): 0 \le x \le 3, 0 \le y \le \min\{x^2 + 2, 2x + 2\}\}$ be A. Then 12A is equal to _____.

Ans. **164**

Sol.



$$A = \int_0^2 (x^2 + 2)dx + \int_2^3 (2x + 2)dx$$

$$A = \frac{41}{3}$$

$$12A = 41 \times 4 = 164$$

26. Let O be the origin, and M and N be the points on the lines $\frac{x-5}{4} = \frac{y-4}{1} = \frac{z-5}{3}$ and $\frac{x+8}{12} = \frac{y+2}{5} = \frac{z+11}{9}$ respectively such that MN is the shortest distance between the given lines. Then $\overrightarrow{OM} \cdot \overrightarrow{ON}$ is equal to _____.

Ans. **9**

Sol. $L_1 : \frac{x-5}{4} = \frac{y-4}{1} = \frac{z-5}{3} = \lambda$ drs $(4,1,3) = b_1$

M $(4\lambda+5, \lambda + 4, 3\lambda + 5)$

$$L_2 : \frac{x+8}{12} = \frac{y+2}{5} = \frac{z+11}{9} = \mu$$

N$(12\mu - 8, 5\mu - 2, 9\mu - 11)$

$$\overrightarrow{MN} = (4\lambda - 12\mu + 13, \lambda - 5\mu + 6, 3\lambda - 9\mu + 16) \quad ..(1)$$

Now

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 1 & 3 \\ 12 & 5 & 9 \end{vmatrix} = -6\hat{i} + 8\hat{k} \quad \ldots(2)$$

Equation (1) and (2)

$$\therefore \frac{4\lambda - 12\mu + 13}{-6} = \frac{\lambda - 5\mu + 6}{0} = \frac{3\lambda - 9\mu + 16}{8}$$

I and II

$$\lambda - 5\mu + 6 = 0 \quad \ldots.(3)$$

I and III

$$\lambda - 3\mu + 4 = 0 \quad \ldots.(4)$$

Solve (3) and (4) we get

$\lambda = -1, \mu = 1$

$\therefore$ M (1, 3, 2)

N (4, 3, –2)

$\therefore$ $\overrightarrow{OM} \cdot \overrightarrow{ON} = 4 + 9 - 4 = 9$

27. Let

$$f(x) = \sqrt{\lim_{r \to x}\left\{\frac{2r^2\left[(f(r))^2 - f(x)f(r)\right]}{r^2 - x^2} - r^3 e^{\frac{f(r)}{r}}\right\}}$$ be

differentiable in $(-\infty, 0) \cup (0, \infty)$ and $f(1) = 1$. Then the value of ea, such that $f(a) = 0$, is equal to _____.

Ans. **2**

Sol. $f(1)=1, \; f(a) = 0$

$$f^2(x) = \lim_{r \to x}\left(\frac{2r^2(f^2(r) - f(x)f(r))}{r^2 - x^2} - r^3 e^{\frac{f(r)}{r}}\right)$$

$$= \lim_{r \to x}\left(\frac{2r^2 f(r)}{r + x}\frac{(f(r) - f(x))}{r - x} - r^3 e^{\frac{f(r)}{r}}\right)$$

$$f^2(x) = \frac{2x^2 f(x)}{2x} f'(x) - x^3 e^{\frac{f(x)}{x}}$$

$$y^2 = xy\frac{dy}{dx} - x^3 e^{\frac{y}{x}}$$

$$\frac{y}{x} = \frac{dy}{dx} - \frac{x^2}{y} e^{\frac{y}{x}}$$

Put $\boxed{y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}}$

$$v = v + x\frac{dv}{dx} - \frac{x}{v}e^{v}$$

$$\frac{dv}{dx} = \frac{e^{v}}{v} \Rightarrow e^{-v}\,vdv = dx$$

Integrating both side

$e^{v}(x + c) + 1 + v = 0$

$f(1) = 1 \Rightarrow x = 1, y = 1$

$$\Rightarrow c = -1 - \frac{2}{e}$$

$$e^{v}\left(-1 - \frac{2}{e} + x\right) + 1 + v = 0$$

$$e^{\frac{y}{x}}\left(-1 - \frac{2}{e} + x\right) + 1 + \frac{y}{x} = 0$$

$$x = a,\ y = 0 \Rightarrow a = \frac{2}{e}$$

$ae = 2$

28. Remainder when $64^{32^{32}}$ is divided by 9 is equal to ____.

Ans. 1

Sol. Let $32^{32} = t$

$64^{32^{32}} = 64^{t} = 8^{2t} = (9 - 1)^{2t}$

$= 9k + 1$

Hence remainder = 1

29. Let the set $C = \{(x, y) \mid x^2 - 2^y = 2023, x, y \in \mathbb{N}\}$.

Then $\sum_{(x,y)\in C}(x + y)$ is equal to ______.

Ans. 46

Sol. $x^2 - 2^y = 2023$

$\Rightarrow \boxed{x = 45, y = 1}$

$$\sum_{(x,y)\in C}(x + y) = 46.$$

30. Let the slope of the line 45x + 5y + 3 = 0 be $27r_1 + \frac{9r_2}{2}$ for some r_1, $r_2 \in R$. Then

$$\lim_{x\to 3}\left(\int_3^x \frac{8t^2}{\frac{3r_2x}{2} - r_2x^2 - r_1x^3 - 3x}dt\right)$$ is equal to ____.

Ans. 12

Sol. According to the question ,

$$27r_1 + \frac{9r_2}{2} = -9$$

$$\lim_{x\to 3}\frac{\int_3^x 8t^2dt}{\frac{3r_2x}{2} - r_2x^2 - r_1x^3 - 3x}$$

$$= \lim_{x\to 3}\frac{8x^2}{\frac{3r_2}{2} - 2r_2x - 3r_1x^2 - 3} \quad \text{(using LH' Rule)}$$

$$= \frac{72}{\frac{3r_2}{2} - 6r_2 - 27r_1 - 3}$$

$$= \frac{72}{-\frac{9r_2}{2} - 27r_1 - 3}$$

$$= \frac{72}{9 - 3} = 12$$