FINAL JEE–MAIN EXAMINATION – APRIL, 2024

(Held On Friday 05th April, 2024) **TIME : 3 : 00 PM to 6 : 00 PM**

MATHEMATICS **TEST PAPER WITH SOLUTION**

SECTION-A

1. Let $f: [-1, 2] \to R$ be given by $f(x) = 2x^2 + x + [x^2] - [x]$, where [t] denotes the greatest integer less than or equal to t. The number of points, where f is not continuous, is :

(1) 6 (2) 3

(3) 4 (4) 5

Ans. (3)

Sol. Doubtful points : $-1, 0, 1, \sqrt{2}, \sqrt{3}, 2$

at $x = \sqrt{2}, \sqrt{3}$

$$f(x) = \underset{\text{Cont.}}{\left(2x^2 + x - [x]\right)} + \underset{\text{Cont.}}{\left[x^2\right]} = \text{Discount}$$

at $x = -1$:

$$\left.\begin{array}{l}\text{RHL} \Rightarrow f(x) = (2-1-(-1)) + 0 = 2\\ f(-1) = 2 - 1 - (-1) + 1 = 3\end{array}\right\}\text{Dis.}$$

at $x = 2$:

$$\left.\begin{array}{l}\text{LHL} \Rightarrow f(x) = 8 + 2 - 1 + 3 = 12\\ f(2) = 8 + 2 - 2 + 4 = 12\end{array}\right\}\text{Cont.}$$

at $x = 0$:

$$\left.\begin{array}{l}\text{LHL} \Rightarrow 0 + 0 - (-1) + 0 = 1\\ f(0) = 0\end{array}\right\}\text{Dis.}$$

at $x = 1$

$$\left.\begin{array}{l}\text{LHL} \Rightarrow 2 + 1 - 0 + 0 = 3\\ f(1) = 3 - 1 + 1 = 3\\ \text{RHL} \Rightarrow 2 + 1 - 1 + 1 = 3\end{array}\right\}\text{Cont.}$$

2. The differential equation of the family of circles passing the origin and having center at the line $y = x$ is :

(1) $(x^2 - y^2 + 2xy)dx = (x^2 - y^2 + 2xy)dy$

(2) $(x^2 + y^2 + 2xy)dx = (x^2 + y^2 - 2xy)dy$

(3) $(x^2 - y^2 + 2xy)dx = (x^2 - y^2 - 2xy)dy$

(4) $(x^2 + y^2 - 2xy)dx = (x^2 + y^2 + 2xy)dy$

Ans. (3)

Sol. $C \equiv x^2 + y^2 + gx + gy = 0$ .....(1)

$2x + 2yy' + g + gy' = 0$

$$g = -\left(\frac{2x + 2yy'}{1 + y'}\right)$$

Put in (1)

$$x^2 + y^2 - \left(\frac{2x + 2yy'}{1 + y'}\right)(x + y) = 0$$

$(x^2 - y^2 - 2xy)y' = x^2 - y^2 + 2xy$

3. Let $S_1 = \{z \in C : |z| \le 5\}$,

$$S_2 = \left\{z \in C : \text{Im}\left(\frac{z + 1 - \sqrt{3}i}{1 - \sqrt{3}i}\right) \ge 0\right\} \text{ and}$$

$S_3 = \{z \in C : \text{Re}(z) \ge 0\}$. Then

(1) $\frac{125\pi}{6}$ (2) $\frac{125\pi}{24}$

(3) $\frac{125\pi}{4}$ (4) $\frac{125\pi}{12}$

Ans. (4)

Sol. $S_1 : x^2 + y^2 \le 25$ .....(1)

$$S_2 : \text{Im of } \frac{z + \left(1 - \sqrt{3}i\right)}{\left(1 - \sqrt{3}i\right)} \ge 0$$

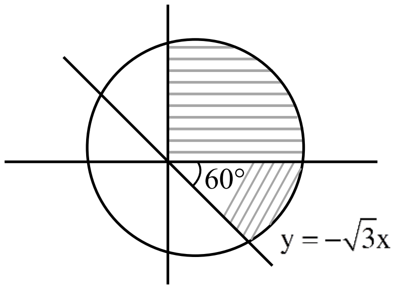
$$\text{Im of } \left(\frac{x + iy}{1 - \sqrt{3}i} + 1\right) \ge 0$$

$$\text{Im of } \left(\frac{(x + iy)\left(1 + \sqrt{3}i\right)}{4}\right) \ge 0$$

$\Rightarrow \sqrt{3}x + y \ge 0$ ......(2)

$S_3 : x \ge 0$ ......(3)

$$\text{Area} = \frac{5}{12}\left(\pi(5)^2\right)$$



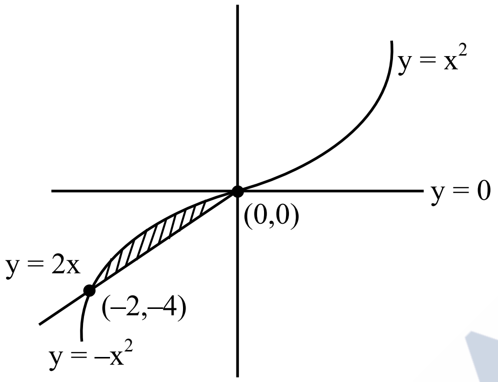
4. The area enclosed between the curves y = x|x| and y = x – |x| is :

(1) $\frac{8}{3}$ (2) $\frac{2}{3}$

(3) 1 (4) $\frac{4}{3}$

Ans. (4)

Sol.



$$A=\int_{-2}^{0} -x^2-2x \;=\frac{4}{3}$$

5. 60 words can be made using all the letters of the word BHBJO, with or without meaning. If these words are written as in a dictionary, then the 50^{th} word is :

(1) OBBHJ (2) HBBJO

(3) OBBJH (4) JBBOH

Ans. (3)

Sol. B B H J O

B ____ $4!=24$

H ____ $\frac{4!}{2!}=12$

J ____ $\frac{4!}{2!}=12$

O B B H J

O B B J H $\rightarrow 50^{th}$ rank

6. Let $\vec{a}=2\hat{i}+5\hat{j}-\hat{k}$, $\vec{b}=2\hat{i}-2\hat{j}+2\hat{k}$ and $\vec{c}$ be three vectors such that $(\vec{c}+\hat{i})\times(\vec{a}+\vec{b}+\hat{i})=\vec{a}\times(\vec{c}+\hat{i})$. $\vec{a}.\vec{c}=-29$, then $\vec{c}.(-2\hat{i}+\hat{j}+\hat{k})$ is equal to :

(1) 10 (2) 5

(3) 15 (4) 12

Ans. (2)

Sol. Let's assume $\vec{v}=\vec{a}+\vec{b}+\hat{i}$

$=5\hat{i}+3\hat{j}+\hat{k}$

and $\vec{c}+\hat{i}=\vec{p}$

So,

$\vec{p}\times\vec{v}=\vec{a}\times\vec{p}$

$\vec{p}\times\vec{v}+\vec{p}\times\vec{a}=\vec{0}$

$\vec{p}\times(\vec{v}+\vec{a})=\vec{0}$

$\Rightarrow \vec{p}=\lambda(\vec{v}+\vec{a})$

$\vec{c}+\mathrm{i}=\lambda(7\hat{i}+8\hat{j})$

$\vec{a}.\vec{c}+\vec{a}.\hat{i}=\lambda\vec{a}.(7\hat{i}+8\hat{j})$

$-29+2=\lambda(14+40)$

$\lambda=-\frac{1}{2}$

$\vec{c}.(-2\hat{i}+\hat{j}+\hat{k})+\hat{i}.(-2\hat{i}+\hat{j}+\hat{k})=\lambda(7\hat{i}+8\hat{j}).(-2\hat{i}+\hat{j}+\hat{k})$

$=-\frac{1}{2}(-14+8)+2=5$

7. Consider three vectors $\vec{a},\vec{b},\vec{c}$. Let $|\vec{a}|=2,|\vec{b}|=3$ and $\vec{a}=\vec{b}\times\vec{c}$. If $\alpha\in\left[0,\frac{\pi}{3}\right]$ is the angle between the vectors $\vec{b}$ and $\vec{c}$, then the minimum value of $27|\vec{c}-\vec{a}|^2$ is equal to :

(1) 110 (2) 105

(3) 124 (4) 121

Ans. (3)

Sol. $|\vec{c}-\vec{a}| = |\vec{c}|^2 + |\vec{a}|^2 - 2\vec{a}.\vec{c}$

$= |\vec{c}|^2 + 4 - 0$

$\because \ \vec{a} = \vec{b} \times \vec{c}$

$|\vec{a}| = |\vec{b} \times \vec{c}|$

$2 = 3|\vec{c}|\sin\alpha$

$|\vec{c}| = \frac{2}{3}\text{cosec}\alpha \qquad \alpha \in \left[0, \frac{\pi}{3}\right]$

$|\vec{c}|_{min} = \frac{2}{3} \times \frac{2}{\sqrt{3}} \qquad \text{cosec}\alpha \in \left[\frac{2}{\sqrt{3}}, \infty\right)$

$\Rightarrow 27|\vec{c}-\vec{a}|^2_{min} = 27\left(\frac{16}{27} + 4\right) = 124$

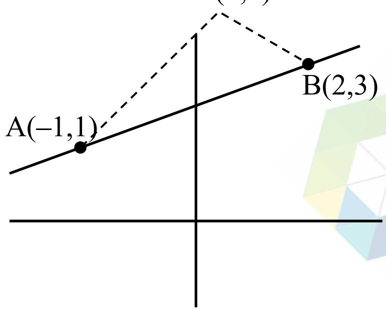
8. Let A(–1, 1) and B(2, 3) be two points and P be a variable point above the line AB such that the area of ΔPAB is 10. If the locus of P is ax + by = 15, then 5a + 2b is :

(1) $-\frac{12}{5}$ (2) $-\frac{6}{5}$

(3) 4 (4) 6

Ans. (1)

Sol.



$\frac{1}{2}\begin{vmatrix} h & k & 1 \\ -1 & 1 & 1 \\ 2 & 3 & 1 \end{vmatrix} = 10$

$-2x + 3y = 25$

$-\frac{6}{5}x + \frac{9}{5}y = 15$

$a = -\frac{6}{5}, b = \frac{9}{5}$

$5a = -6, 2b = \frac{18}{5}$

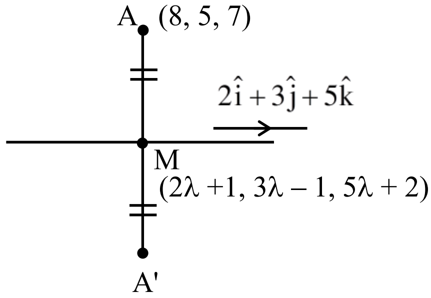
9. Let (α, β, γ) be the point (8, 5, 7) in the line $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-2}{5}$. Then $\alpha + \beta + \gamma$ is equal to

(1) 16 (2) 18

(3) 14 (4) 20

Ans. (3)

Sol.



$\overrightarrow{AM}.(2\hat{i}+3\hat{j}+5\hat{k}) = 0$

$(2\lambda - 7)(2) + (3\lambda - 6)(3) + (5\lambda - 5)(5) = 0$

$38\lambda = 57$

$\lambda = \frac{3}{2}$

$M\left(4, \frac{7}{2}, \frac{19}{2}\right)$

A'(0,2,12)

10. If the constant term in the expansion of $\left(\frac{\sqrt[5]{3}}{x} + \frac{2x}{\sqrt[3]{5}}\right)^{12}$, $x \neq 0$, is $\alpha \times 2^8 \times \sqrt[5]{3}$, then 25α is equal to :

(1) 639 (2) 724

(3) 693 (4) 742

Ans. (3)

Sol. $T_{r+1} = {}^{12}C_r\left(\frac{3^{1/5}}{x}\right)^{12-r}\left(\frac{2x}{5^{1/3}}\right)^r$

$T_{r+1} = \frac{{}^{12}C_r(3)^{\frac{12-r}{5}}(2)^r(x)^{2r-12}}{(5)^{r/3}}$

$r = 6$

$T_7 = \frac{{}^{12}C_6(3)^{6/5}(2)^6}{5^2} = \left(\frac{9 \times 11 \times 7}{25}\right)2^8.3^{1/5}$

$25\alpha = 693$

11. Let f, g : R → R be defined as : $f(x) = |x - 1|$ and $g(x) = \begin{cases} e^x, & x \ge 0 \\ x+1, & x \le 0 \end{cases}$. Then the function $f(g(x))$ is

(1) neither one-one nor onto.

(2) one-one but not onto.

(3) both one-one and onto.

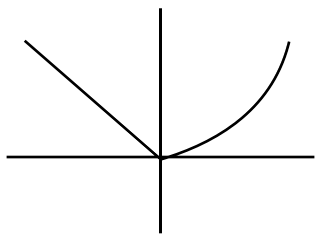
(4) onto but not one-one.

Ans. (1)

Sol. $f(g(x)) = |g(x) - 1|$

$$fog\begin{bmatrix} |e^x - 1| & x \ge 0 \\ |x+1-1| & x \le 0 \end{bmatrix}$$

$$fog\begin{bmatrix} e^x - 1 & x \ge 0 \\ -x & x \le 0 \end{bmatrix}$$



12. Let the circle $C_1 : x^2 + y^2 - 2(x + y) + 1 = 0$ and C_2 be a circle having centre at (–1, 0) and radius 2. If the line of the common chord of C_1 and C_2 intersects the y-axis at the point P, then the square of the distance of P from the centre of C_1 is :

(1) 2 (2) 1

(3) 6 (4) 4

Ans. (1)

Sol. $S_1 : x^2 + y^2 - 2x - 2y + 1 = 0$

$S_2 : x^2 + y^2 + 2x - 3 = 0$

Common chord = $S_1 - S_2 = 0$

$-4x - 2y + 4 = 0$

$2x + y = 2 \Rightarrow P(0, 2)$

$d^2_{(c,p)} = (1 - 0)^2 + (2 - 1)^2 = 2$

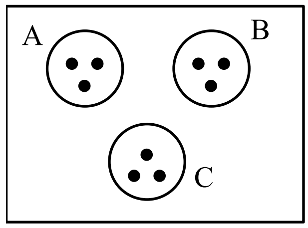
13. Let the set S = {2, 4, 8, 16, ..., 512} be partitioned into 3 sets A, B, C with equal number of elements such that $A \cup B \cup C = S$ and $A \cap B = B \cap C = A \cap C = \phi$. The maximum number of such possible partitions of S is equal to :

(1) 1680 (2) 1520

(3) 1710 (4) 1640

Ans. (1)

Sol.



$$\frac{9!}{(3!3!3!)3!} \times 3!$$

14. The values of m, n, for which the system of equations

$x + y + z = 4,$

$2x + 5y + 5z = 17,$

$x + 2y + mz = n$

has infinitely many solutions, satisfy the equation :

(1) $m^2 + n^2 - m - n = 46$

(2) $m^2 + n^2 + m + n = 64$

(3) $m^2 + n^2 + mn = 68$

(4) $m^2 + n^2 - mn = 39$

Ans. (4)

Sol. $D = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 5 & 5 \\ 1 & 2 & m \end{vmatrix} = 0 \Rightarrow m = 2$

$D_3 = \begin{vmatrix} 1 & 1 & 4 \\ 2 & 5 & 17 \\ 1 & 2 & n \end{vmatrix} = 0 \Rightarrow n = 7$

15. The coefficients a, b, c in the quadratic equation $ax^2 + bx + c = 0$ are from the set {1, 2, 3, 4, 5, 6}. If the probability of this equation having one real root bigger than the other is p, then 216p equals :

(1) 57 (2) 38

(3) 19 (4) 76

Ans. (2)

Sol. $D > 0$

$b^2 > 4ac$

$b = 3 : (a, c) = (1, 1)(1, 2)(2,1)$

$b = 4 : (a, c) = (1, 1)(1, 2)(2,1)(1,3)(3,1)$

$b = 5 : (a, c) = (1,1)(1,2)(2,1)\ (1,3)(3,1)(1,4)(4,1)$
$(1,5)(5,1)(1,6)(6,1)(2,3)(3,2)(2,2)$

$b = 6 : (a, c) = (1,1)(1,2)(2,1)\ (1,3)(3,1)(1,4)(4,1)$
$(1,5)(5,1)(1,6)(6,1)(2,3)(3,2)(2,4)(4,2)(2,2)$

fav. cases = 38

Prob. : $\dfrac{38}{6\times6\times6}$

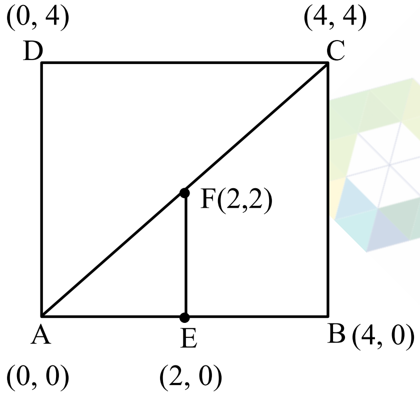
16. Let ABCD and AEFG be squares of side 4 and 2 units, respectively. The point E is on the line segment AB and the point F is on the diagonal AC. Then the radius r of the circle passing through the point F and touching the line segments BC and CD satisfies :

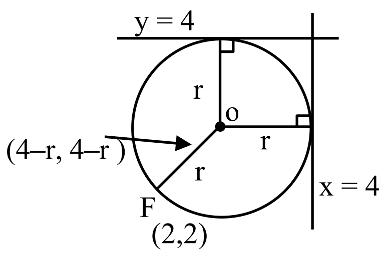
(1) $r = 1$ (2) $r^2 - 8r + 8 = 0$

(3) $2r^2 - 4r + 1 = 0$ (4) $2r^2 - 8r + 7 = 0$

Ans. (2)

Sol.





$OF^2 = r^2$

$(2 - r)^2 + (2 - r)^2 = r^2$

$r^2 - 8r + 8 = 0$

17. Let $\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$, m, n > 0. If $\int_0^1 (1-x^{10})^{20}dx = a\times\beta(b,c)$, then 100(a + b + x) equals ______.

(1) 1021 (2) 1120

(3) 2012 (4) 2120

Ans. (4)

Sol. $I = \int_0^1 1.\left(1-x^{10}\right)^{20} dx$

$x^{10} = t$

$x = t^{1/10}$

$dx = \dfrac{1}{10}(t)^{-9/10} dt$

$I = \int_0^1 (1-t)^{20} \dfrac{1}{10}(t)^{-9/10} dt$

$I = \dfrac{1}{10}\int_0^1 t^{-9/10}(1-t)^{20} dt$

$a = \dfrac{1}{10}$ $b = \dfrac{1}{10}$ $c = 21$

18. Let $\alpha\beta \neq 0$ and $A = \begin{bmatrix} \beta & \alpha & 3 \\ \alpha & \alpha & \beta \\ -\beta & \alpha & 2\alpha \end{bmatrix}$.

If $B = \begin{bmatrix} 3\alpha & -9 & 3\alpha \\ -\alpha & 7 & -2\alpha \\ -2\alpha & 5 & -2\beta \end{bmatrix}$ is the matrix of cofactors of the elements of A, then det(AB) is equal to :

(1) 343 (2) 125

(3) 64 (4) 216

Ans. (4)

Sol. Equating co-factor fo A_{21}

$(2\alpha^2 - 3\alpha) = \alpha$

$\alpha = 0, 2$ (accept)

Now, $2\alpha^2 - \alpha\beta = 3\alpha$

$\alpha = 2$ $\beta = 1$

$|AB| = |A\ \text{cof}\ (A)| = |A|^3$

$A = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 2 & 1 \\ -1 & 2 & 4 \end{vmatrix} = 6 - 2(9) + 3(6) = 6$

19. If $y(\theta)=\dfrac{2\cos\theta+\cos 2\theta}{\cos 3\theta+4\cos 2\theta+5\cos\theta+2}$,

then at $\theta=\dfrac{\pi}{2}$, $y''+y'+y$ is equal to:

(1) $\dfrac{3}{2}$ (2) 1

(3) $\dfrac{1}{2}$ (4) 2

Ans. (4)

Sol. $y=\dfrac{2\cos\theta+2\cos^2\theta-1}{4\cos^3\theta-3\cos\theta+8\cos^2\theta-4+5\cos\theta+2}$

$y=\dfrac{\left(2\cos^2\theta+2\cos\theta-1\right)}{\left(2\cos^2\theta+2\cos\theta-1\right)\left(2\cos\theta+2\right)}$

$y=\dfrac{1}{2}\left(\dfrac{1}{1+\cos\theta}\right)$

$\Rightarrow \theta=\dfrac{\pi}{2} \quad y=\dfrac{1}{2}$

$y'=\dfrac{1}{2}\left(\dfrac{-1}{(1+\cos\theta)^2}\times(-\sin\theta)\right)$

$\Rightarrow \theta=\dfrac{\pi}{2} \quad y=\dfrac{1}{2}$

$y''=\dfrac{1}{2}\left[\dfrac{\cos\theta(1+\cos\theta)^2-\sin\theta(2)(1+\cos\theta)(-\sin\theta)}{(1+\cos\theta)^4}\right]$

$\Rightarrow \theta=\dfrac{\pi}{2} \qquad y=1$

20. For $x\ge 0$, the least value of K, for which $4^{1+x}+4^{1-x}$, $\dfrac{K}{2}$, 16^x+16^{-x} are three consecutive terms of an A.P. is equal to :

(1) 10 (2) 4

(3) 8 (4) 16

Ans. (1)

Sol. $k=4\underbrace{\left(4^x+\dfrac{1}{4^x}\right)}_{\ge 2}+\underbrace{\left(4^{2x}+\dfrac{1}{4^{2x}}\right)}_{\ge 2}$

$k\ge 10$

SECTION-B

21. Let the mean and the standard deviation of the probability distribution

X	α	1	0	–3
P(X)	$\frac{1}{3}$	K	$\frac{1}{6}$	$\frac{1}{4}$

be μ and σ, respectively. If $\sigma-\mu=2$, then $\sigma+\mu$ is equal to _____.

Ans. (5)

Sol. $\dfrac{1}{3}+k+\dfrac{1}{6}+\dfrac{1}{4}=1 \qquad \Rightarrow k=\dfrac{1}{4}$

$\mu=\dfrac{\alpha}{3}+\dfrac{1}{4}-\dfrac{3}{4}$

$\boxed{\mu=\dfrac{\alpha}{3}-\dfrac{1}{2}}$

$\sigma=\sqrt{\left(\alpha^2\dfrac{1}{3}+\dfrac{1}{4}+9\dfrac{1}{4}\right)-\left(\dfrac{\alpha}{3}-\dfrac{1}{2}\right)^2}$

$\sigma=\sqrt{\dfrac{2\alpha^2}{9}+\dfrac{\alpha}{3}+\dfrac{9}{4}}$

$\sigma=\mu+2$

$\sigma^2=(\mu+2)^2 \Rightarrow \dfrac{2\alpha^2}{9}+\dfrac{\alpha}{3}+\dfrac{9}{4}=\dfrac{\alpha^2}{9}+\dfrac{9}{4}+\alpha$

$\dfrac{\alpha^2}{9}-\dfrac{2\alpha}{3}=0$

$\alpha=0$, (reject) or $\alpha=6$

($\because \ x=0$ is already given)

$\Rightarrow \sigma+\mu=2\mu+2$

$=5$

22. Let $y=y(x)$ be the solution of the differential equation $\dfrac{dy}{dx}+\dfrac{2x}{(1+x^2)^2}y=xe^{\frac{1}{(1+x^2)}}$; $y(0)=0$. Then the area enclosed by the curve $f(x)=y(x)e^{-\frac{1}{(1+x^2)}}$ and the line $y-x=4$ is ____.

Ans. (18)

Sol. IF $= e^{\int \frac{2x}{(1+x^2)^2}dx} = e^{\frac{-1}{1+x^2}}$

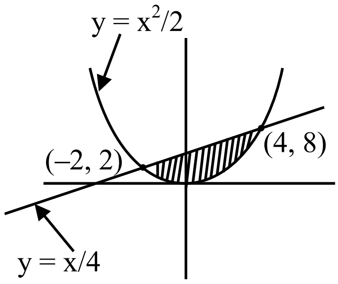
$y \cdot e^{\frac{-1}{1+x^2}} = \int x \cdot e^{\frac{1}{1+x^2}} \cdot e^{\frac{-1}{1+x^2}} dx$

$y \cdot e^{\frac{-1}{1+x^2}} = \frac{x^2}{2} + c$

$(0, 0) \Rightarrow \boxed{C = 0}$

$y(x) = \frac{x^2}{2} e^{\frac{1}{1+x^2}}$

$f(x) = \frac{x^2}{2}$



$A = \int_{-2}^{4} (x+4) - \frac{x^2}{2} dx = 18$

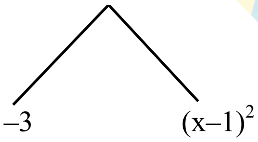
23. The number of solutions of $\sin^2 x + (2 + 2x - x^2)\sin x - 3(x-1)^2 = 0$, where $-\pi \le x \le \pi$, is

Ans. (2)

Sol. $\sin^2 x - (x^2 - 2x - 2)\sin x - 3(x-1)^2 = 0$

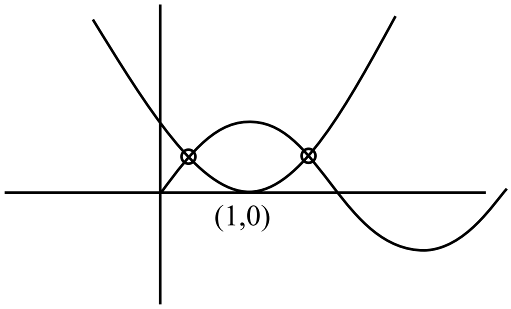
$\sin^2 x - (x-1)^2)\sin x - 3(x-1)^2 = 0$

roots :



$\sin x = -3$ (reject) or $(x-1)^2$

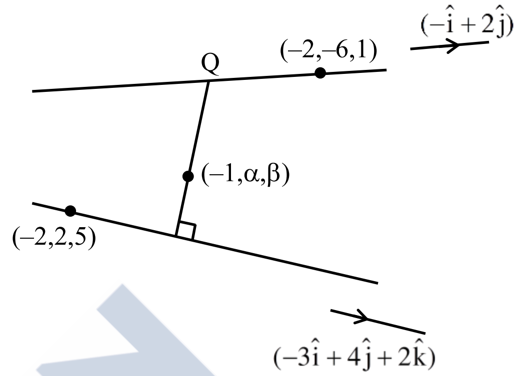
$\sin x = (x-1)^2$



24. Let the point $(-1, \alpha, \beta)$ lie on the line of the shortest distance between the lines $\frac{x+2}{-3} = \frac{y-2}{4} = \frac{z-5}{2}$ and $\frac{x+2}{-1} = \frac{y+6}{2} = \frac{z-1}{0}$. Then $(\alpha - \beta)^2$ is equal to _____.

Ans. (25)

Sol.



$P(-3\lambda - 2, 4\lambda + 2, 2\lambda + 5)$

$Q(-\mu - 2, 2\mu - 6, 1)$

DRS of PQ $= (3\lambda - \mu, 2\mu - 4\lambda - 8, -2\lambda - 4)$

DRS of PQ $= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 2 & 0 \\ -3 & 4 & 2 \end{vmatrix}$

$= \left(4\hat{i} + 2\hat{j} + 2\hat{k}\right)$

OR

(2, 1, 1)

$\frac{3\lambda - \mu}{2} = \frac{2\mu - 4\lambda - 8}{1} = \frac{-2\lambda - 4}{1}$

$\Rightarrow \mu = \lambda + 2 \ \& \ 7\lambda = \mu - 8$

$\boxed{\lambda = -1}$ $\boxed{\mu = 1}$

Q : (–3, –4, 1)

$L_{PQ} = \frac{x+3}{2} = \frac{y+4}{1} = \frac{z-1}{1}$

$(-1, \alpha, \beta) \Rightarrow 1 = \frac{\alpha + 4}{1} = \frac{\beta - 1}{1}$

$\Rightarrow \alpha = -3, \beta = 2$

$(\alpha - \beta)^2 = 25$

25. If

$$1+\frac{\sqrt{3}-\sqrt{2}}{2\sqrt{3}}+\frac{5-2\sqrt{6}}{18}+\frac{9\sqrt{3}-11\sqrt{2}}{36\sqrt{3}}+\frac{49-20\sqrt{6}}{180}+....$$

upto $\infty = 2\left(\sqrt{\frac{b}{a}}+1\right)\log_e\left(\frac{a}{b}\right)$, where a and b are integers with gcd(a, b) = 1, then 11a + 18b is equal to ______.

Ans. (76)

Sol. $S=1+\frac{x}{2\sqrt{3}}+\frac{x^2}{18}+\frac{x^3}{36\sqrt{3}}+\frac{x^4}{180}+....\infty$

Put $\frac{x}{\sqrt{3}}=t$, where $x=\sqrt{3}-\sqrt{2}$

$S=1+\frac{t}{2}+\frac{t^2}{6}+\frac{t^3}{12}+\frac{t^4}{20}+....$

$S=1+t\left(1-\frac{1}{2}\right)+t^2\left(\frac{1}{2}-\frac{1}{3}\right)+t^3\left(\frac{1}{3}-\frac{1}{4}\right)+t^4\left(\frac{1}{4}-\frac{1}{5}\right)$

$S=\left(1+t+\frac{t^2}{2}+\frac{t^3}{3}+\frac{t^3}{4}+...\right)-\left(\frac{t}{2}+\frac{t^2}{3}+\frac{t^3}{4}+\frac{t^4}{5}+...\right)$

$S=\left(t+\frac{t^2}{2}+...\right)-\frac{1}{t}\left(t+\frac{t^2}{2}+\frac{t^3}{3}+...\right)+2$

$S=2+\left(1-\frac{1}{t}\right)(-\log(1-t))=\left(\frac{1}{t}-1\right)\log(1-t)+2$

$S=2+\left(\frac{\sqrt{3}}{\sqrt{3}-\sqrt{2}}-1\right)\log\left(1-\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}}\right)$

$S=2+\left(\frac{\sqrt{2}}{\sqrt{3}-\sqrt{2}}\right)\log e\frac{\sqrt{2}}{\sqrt{3}}$

$S=2+\frac{(\sqrt{6}+2)}{2}\log e\frac{2}{3}=2+\left(\sqrt{\frac{3}{2}}+1\right)\log e\frac{2}{3}$

a = 2, b = 3

11a + 18b = 11 × 2 + 18 × 3 = 76

26. Let a > 0 be a root of the equation $2x^2 + x - 2 = 0$.

If $\lim\limits_{x\to\frac{1}{a}}\frac{16\left(1-\cos(2+x-2x^2)\right)}{(1-ax^2)}=\alpha+\beta\sqrt{17}$, where $\alpha, \beta \in Z$ then $\alpha + \beta$ is equal to ______.

Ans. (170)

Sol. $2x^2+x-2=0$ with roots a and b

$2x^2-x-2=0$ with roots $\frac{1}{a}$ and $\frac{1}{b}$

$\lim\limits_{x\to\frac{1}{a}} 16\cdot\frac{\left(1-\cos 2\left(x-\frac{1}{a}\right)\left(x-\frac{1}{b}\right)\right)}{4\left(x-\frac{1}{b}\right)^2}\times\frac{4\left(x-\frac{1}{b}\right)^2}{a^2\left(x-\frac{1}{a}\right)^2}$

$=16\times\frac{2}{a^2}\left(\frac{1}{a}-\frac{1}{b}\right)^2$

$=\frac{32}{a^2}\left(\frac{17}{4}\right)=\frac{17.8}{a^2}=\frac{17\times8\times16}{\left(-1+\sqrt{117}\right)^2}$

$=\frac{136.16}{18.2\sqrt{7}}\times\frac{18+2\sqrt{7}}{18+2\sqrt{7}}$

$=\frac{136}{256}\left(18+2\sqrt{7}\right)\cdot16$

$=153+17\sqrt{17}=\alpha+\beta\sqrt{17}$

$\alpha + \beta = 153 + 17 = 170$

27. If $f(t)=\int_0^{\pi}\frac{2xdx}{1-\cos^2 t\sin^2 x}$, $0 < t < \pi$, then the value of $\int_0^{\frac{\pi}{2}}\frac{\pi^2 dt}{f(t)}$ equals ______.

Ans. (1)

Sol. $f(t)=\int_0^{\pi}\frac{2x}{1-\cos^2 t\sin^2 x}dx$ .....(1)

$$= 2\int_0^{\pi} \frac{(\pi - x)dx}{1 - \cos^2 t \sin^2 x} \quad \text{....(2)}$$

$$2f(t) = 2\int_0^{\pi} \frac{\pi}{1 - \cos^2 t \sin^2 x} dx$$

$$f(t) = \int_0^{\pi} \frac{\pi}{1 - \cos^2 t \sin^2 x} dx$$

divide & by $\cos^2 x$

$$f(t) = \pi \int_0^{\pi} \frac{\sec^2 x dx}{\sec^2 x - \cos^2 t \tan^2 x}$$

$$f(t) = 2\pi \int_0^{\pi/2} \frac{\sec^2 x dx}{\sec^2 x - \cos^2 t \tan^2 x}$$

$\tan x = z$

$\sec^2 x dx = dz$

$$f(t) = 2\pi \int_0^{\infty} \frac{dz}{1 + \sin^2 t \cdot z^2}$$

$$= \frac{\pi^2}{\sin t}$$

Then $\displaystyle\int_0^{\pi/2} \frac{\pi^2}{f(t)} dt$

$$= \int_0^{\pi/2} \sin t \, dt$$

$= 1$

28. Let the maximum and minimum values of $\left(\sqrt{8x - x^2 - 12} - 4\right)^2 + (x - 7)^2$, $x \in R$ be M and m respectively. Then $M^2 - m^2$ is equal to _____.

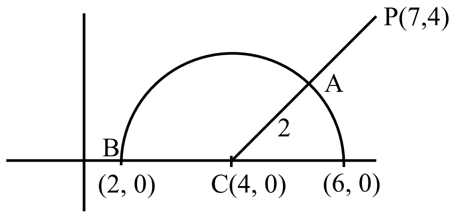
Ans. (1600)

Sol. $(x - 7)^2 + (y - 4)^2$

$y = \sqrt{8x - x^2 - 12}$

$y^2 = -(x - 4)^2 + 16 - 12$

$(x - 4)^2 + y^2 = 4$



$m = 9$

$M = 41$

$M^2 - m^2 = 41^2 - 9^2 = 1600$

29. Let a line perpendicular to the line $2x - y = 10$ touch the parabola $y^2 = 4(x - 9)$ at the point P. The distance of the point P from the centre of the circle $x^2 + y^2 - 14x - 8y + 56 = 0$ is ______.

Ans. (10)

Sol. $y^2 = 4(x - 9)$

slope of tangent $= \dfrac{-1}{2}$

Point of contact $P\left(9 + \dfrac{1}{\left(-\frac{1}{2}\right)^2}, \dfrac{2 \times 1}{\frac{-1}{2}}\right)$

$P(13, -4)$

center of circle $C(7, 4)$

distance $CP = \sqrt{(13 - 7)^2 + (-4 - 4)^2}$

$= 10$

30. The number of real solutions of the equation $x|x + 5| + 2|x + 7| - 2 = 0$ is _____.

Ans. (3)

30. The number of real solutions of the equation $x|x + 5| + 2|x + 7| - 2 = 0$ is _____.

Allen Ans. (3)

Sol. **Case I :** $x \geq -5$

$x^2 + 5x + 2x + 12 = 0$

$x^2 + 7x + 12 = 0$

$x = -3, -4$

Case II : $-7 < x < -5$

$-x^2 - 5x + 2x + 14 - 2 = 0$

$-x^2 - 3x + 12 = 0$

$x = \frac{-3 \pm \sqrt{9+48}}{2}$

$= \frac{-3 \pm \sqrt{57}}{2}$

$x = \frac{-3-\sqrt{57}}{2}, \frac{-3+\sqrt{57}}{2}$ (rejected)

Case III : $x \leq -7$

$-x^2 - 5x - 2x - 14 - 2 = 0$

$x^2 + 7x + 16 = 0$

$D = 49 - 64 < 0$

No solutions

No. of solutions = 3