FINAL JEE–MAIN EXAMINATION – APRIL, 2024

(Held On Saturday 06th April, 2024) **TIME : 3 : 00 PM to 6 : 00 PM**

MATHEMATICS — TEST PAPER WITH SOLUTION

SECTION-A

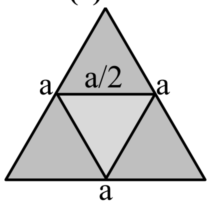
1. Let ABC be an equilateral triangle. A new triangle is formed by joining the middle points of all sides of the triangle ABC and the same process is repeated infinitely many times. If P is the sum of perimeters and Q is be the sum of areas of all the triangles formed in this process, then:

(1) $P^2=36\sqrt{3}Q$ (2) $P^2=6\sqrt{3}Q$

(3) $P=36\sqrt{3}Q^2$ (4) $P^2=72\sqrt{3}Q$

Ans. (1)

Sol.



Area of first $\Delta=\frac{\sqrt{3}a^2}{4}$

Area of second $\Delta=\frac{\sqrt{3}a^2}{4}\frac{a^2}{4}=\frac{\sqrt{3}a^2}{16}$

Area of third $\Delta=\frac{\sqrt{3}a^2}{64}$

sum of area $=\frac{\sqrt{3}a^2}{4}\left(1+\frac{1}{4}+\frac{1}{16}....\right)$

$Q=\frac{\sqrt{3}a^2}{4}\frac{1}{\frac{3}{4}}=\frac{a^2}{\sqrt{3}}$

perimeter of 1^{st} $\Delta = 3a$

perimeter of 2^{nd} $\Delta=\frac{3a}{2}$

perimeter of 3^{rd} $\Delta=\frac{3a}{4}$

$P=3a\left(1+\frac{1}{2}+\frac{1}{4}+...\right)$

$P = 3a.2 = 6a$

$a=\frac{P}{6}$

$Q=\frac{1}{\sqrt{3}}\cdot\frac{P^2}{36}$

$P^2=36\sqrt{3}Q$

2. Let A = {1, 2, 3, 4, 5}. Let R be a relation on A defined by xRy if and only if $4x \le 5y$. Let m be the number of elements in R and n be the minimum number of elements from A × A that are required to be added to R to make it a symmetric relation. Then m + n is equal to:

(1) 24 (2) 23

(3) 25 (4) 26

Ans. (3)

Sol. Given : $4x \le 5y$

then

$R=\{(1,1),(1,2),(1,3),(1,4),(1,5),(2,2),(2,3),(2,4)$
$(2,5),(3,3),(3,4),(3,5),(4,4),(4,5),(5,4),(5,5)\}$

i.e. 16 elements.

i.e. m = 16

Now to make R a symmetric relation add

$\{(2,1)(3,2)(4,3)(3,1)(4,2)(5,3)(4,1)(5,2)(5,1)\}$

i.e. n = 9

So m + n = 25

3. If three letters can be posted to any one of the 5 different addresses, then the probability that the three letters are posted to exactly two addresses is:

(1) $\frac{12}{25}$ (2) $\frac{18}{25}$

(3) $\frac{4}{25}$ (4) $\frac{6}{25}$

Ans. (1)

Sol. Total method $=5^3$

faverable $={}^5C_2\left(2^3-2\right)=60$

probability $=\frac{60}{125}=\frac{12}{25}$

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4. Suppose the solution of the differential equation $\frac{dy}{dx}=\frac{(2+\alpha)x-\beta y+2}{\beta x-2\alpha y-(\beta\gamma-4\alpha)}$ represents a circle passing through origin. Then the radius of this circle is :

(1) $\sqrt{17}$ (2) $\frac{1}{2}$

(3) $\frac{\sqrt{17}}{2}$ (4) 2

Ans. (3)

Sol. $\frac{dy}{dx}=\frac{(2+\alpha)x-\beta y+2}{\beta x-y(2\alpha+\beta)+4\alpha}$

$\beta xdy-(2\alpha+\beta)ydy+4\alpha dy=(2+\alpha)xdx-\beta ydx+2dx$

$\beta(xdy+ydx)-(2\alpha+\beta)ydy+4\alpha dy=(2+\alpha)xdx+2dx$

$\beta xy-\frac{(2\alpha+\beta)y^2}{2}+4\alpha y=\frac{(2+\alpha)x^2}{2}$

$\Rightarrow\ \beta=0$ for this to be circle

$(2+\alpha)\frac{x^2}{2}+\alpha y^2+2x-4\alpha y=0$

coeff. of $x^2=y^2$ $\rangle$ 2 + a = 2a

$\Rightarrow\ \boxed{\alpha=2}$

i.e. $2x^2+2y^2+2x-8y=0$

$x^2+y^2+x-4y=0$

$rd=\sqrt{\frac{1}{4}+4}=\frac{\sqrt{17}}{2}$

5. If the locus of the point, whose distances from the point (2, 1) and (1, 3) are in the ratio 5 : 4, is $ax^2+by^2+cxy+dx+ey+170=0$, then the value of $a^2+2b+3c+4d+e$ is equal to:

(1) 5 (2) –27

(3) 37 (4) 437

Ans. (3)

Sol. let P(x, y)

$\frac{(x-2)^2+(y-1)^2}{(x-1)^2+(y-3)^2}=\frac{25}{16}$

$9x^2+9y^2+14x-118y+170=0$

$a^2+2b+3c+4d+e$

$=81+18+0+56-118$

$=155-118$

$=37$

6. $\lim\limits_{n\to\infty}\frac{(1^2-1)(n-1)+(2^2-2)(n-2)+.....+((n-1)^2-(n-1))\cdot 1}{(1^3+2^3+.....+n^3)-(1^2+2^2+.....+n^2)}$ is equal to:

(1) $\frac{2}{3}$ (2) $\frac{1}{3}$

(3) $\frac{3}{4}$ (4) $\frac{1}{2}$

Ans. (2)

Sol. $\lim\limits_{n\to\infty}\frac{\sum\limits_{r=1}^{n-1}(r^2-r)(n-r)}{\sum\limits_{r=1}^{n}r^3-\sum\limits_{r=1}^{n}r^2}$

$\lim\limits_{n\to\infty}\frac{\sum\limits_{r=1}^{n-1}(-r^3+r^2(n+1)-nr)}{\left(\frac{n(n+1)}{2}\right)^2-\frac{n(n+1)(2n+1)}{6}}$

$\lim\limits_{n\to\infty}\frac{\left(\frac{((n-1)n)}{2}\right)^2+\frac{(n+1)(n-1)n(2n-1)}{6}-\frac{n^2(n-1)}{2}}{\frac{n(n+1)}{2}\left(\frac{n(n+1)}{2}-\frac{2n+1}{3}\right)}$

$\lim\limits_{n\to\infty}\frac{\frac{n(n-1)}{2}\left(\frac{-n(n-1)}{2}+\frac{(n+1)(2n-1)}{3}-n\right)}{\frac{n(n+1)}{2}\frac{3n^2+3n-4n-2}{6}}$

$\lim\limits_{n\to\infty}\frac{(n-1)(-3n^2+3n+2(2n^2+n-1)-6)}{(n+1)(3n^2-n-2)}$

$\lim\limits_{n\to\infty}\frac{(n-1)(n^2+5n-8)}{(n+1)(3n^2-n-2)}=\frac{1}{3}$

7. Let $0\le r\le n$. If ${}^{n+1}C_{r+1}:{}^{n}C_r:{}^{n-1}C_{r-1}=55:35:21$, then $2n+5r$ is equal to:

(1) 60 (2) 62

(3) 50 (4) 55

Ans. (3)

Ans. $\frac{{}^{n+1}C_r}{{}^{n}C_r}=\frac{55}{35}$

$\frac{(n+1)!}{(r+1)!(n-r)}!\frac{r!(n-r)!}{n!}=\frac{11}{7}$

$\frac{(n+1)}{r+1}=\frac{11}{7}$

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$7n = 4 + 11r$

$$\frac{{}^{n}C_r}{{}^{n-1}C_{r-1}} = \frac{35}{21}$$

$$\frac{n!}{r!(n-r)!} = \frac{(r-1)!(n-r)!}{(n-1)!} = \frac{5}{3}$$

$$\frac{n}{r} = \frac{5}{3}$$

$3n = 5r$

By solving $r = 6$ $\quad n = 10$

$2n + 5r = 50$

8. A software company sets up m number of computer systems to finish an assignment in 17 days. If 4 computer systems crashed on the start of the second day, 4 more computer systems crashed on the start of the third day and so on, then it took 8 more days to finish the assignment. The value of m is equal to :

(1) 125 (2) 150

(3) 180 (4) 160

Ans. (2)

Sol. $17m = m + (m-4) + (m - 4 \times 2) ... + ... (m - 4 \times 24)$

$17\,m = 25m - 4\,(1 + 2\,...24)$

$$8m = \frac{4 \cdot 24 \cdot 25}{2} = 150$$

9. If z_1, z_2 are two distinct complex number such that $\left|\dfrac{z_1 - 2z_2}{\frac{1}{2} - z_1\bar{z}_2}\right| = 2$, then

(1) either z_1 lies on a circle of radius 1 or z_2 lies on a circle of radius $\frac{1}{2}$

(2) either z_1 lies on a circle of radius $\frac{1}{2}$ or z_2 lies on a circle of radius 1.

(3) z_1 lies on a circle of radius $\frac{1}{2}$ and z_2 lies on a circle of radius 1.

(4) both z_1 and z_2 lie on the same circle.

Ans. (1)

Sol. $$\frac{z_1 - 2z_2}{\frac{1}{2} - z_1\bar{z}_2} \times \frac{\bar{z}_1 - 2\bar{z}_2}{\frac{1}{2} - \bar{z}_1 z_2} = 4$$

$$|z_1|^2\, 2z_1\bar{z}_2 - 2\bar{z}_1 z_2 + 4|z_2|^2$$

$$= 4\left(\frac{1}{4} - \frac{\bar{z}_1 z_2}{2} - \frac{z_1\bar{z}_2}{2} + |z_1|^2|z_2|^2\right)$$

$$z_1\bar{z}_1 + 2z_2 \cdot 2\bar{z}_2 - z_1\bar{z}_1 2z_2\, 2\bar{z}_2 - 1 = 0$$

$$(z,\bar{z}_1 - 1)(1 - 2z_2 \cdot 2\bar{z}_2) = 0$$

$$(|z_1|^2 - 1)(|2z_2|^2 - 1) = 0$$

10. If the function $f(x) = \left(\frac{1}{x}\right)^{2x}$; $x > 0$ attains the maximum value at $x = \frac{1}{e}$ then :

(1) $e^{\pi} < \pi^{e}$ (2) $e^{2\pi} < (2\pi)^{e}$

(3) $e^{\pi} > \pi^{e}$ (4) $(2e)^{\pi} > \pi^{(2e)}$

Ans. (3)

Sol. Let $y = \left(\frac{1}{x}\right)^{2x}$

$$\ell ny = 2x\ell n\left(\frac{1}{x}\right)$$

$$\ell ny = -2x\ell nx$$

$$\frac{1}{y}\frac{dy}{dx} = -2(1 + \ell nx)$$

for $x > \frac{1}{e}$ f^{n} is decreasing

so, $e < \pi$

$$\left(\frac{1}{e}\right)^{2e} > \left(\frac{1}{\pi}\right)^{2\pi}$$

$e^{\pi} > \pi^{e}$

11. Let $\vec{a} = 6\hat{i} + \hat{j} - \hat{k}$ and $\vec{b} = \hat{i} + \hat{j}$. If $\vec{c}$ is a is vector such that $|\vec{c}| \geq 6$, $\vec{a}.\vec{c} = 6|\vec{c}|$, $|\vec{c} - \vec{a}| = 2\sqrt{2}$ and the angle between $\vec{a} \times \vec{b}$ and $\vec{c}$ is 60°, then $|(\vec{a} \times \vec{b}) \times \vec{c}|$ is equal to:

(1) $\frac{9}{2}(6 - \sqrt{6})$ (2) $\frac{3}{2}\sqrt{3}$

(3) $\frac{3}{2}\sqrt{6}$ (4) $\frac{9}{2}(6 + \sqrt{6})$

Ans. (4)

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Sol. $|(\vec{a}\times\vec{b}\times\vec{c})| = |\vec{a}\times\vec{b}||\vec{c}|\dfrac{\sqrt{3}}{2}$

$|\vec{c}-\vec{a}| = 2\sqrt{2}$

$|c|^2 + |a|^2 - 2\vec{c}\cdot\vec{a} = 8$

$|z|^2 + 38 - 12|z| = 8$

$|z|^2 - 12|z| + 30 = 0$

$|z| = \dfrac{12 \pm \sqrt{144-120}}{2}$

$= \dfrac{12 \pm 2\sqrt{6}}{2}$

$|z| = 6 + \sqrt{6}$

$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{\ell} & \hat{j} & \hat{k} \\ 6 & 1 & -1 \\ 1 & 1 & 0 \end{vmatrix}$

$\hat{\ell} - \hat{j} + 5\hat{k}$

$|\vec{a}\times\vec{b}| = \sqrt{27}$

$|(\vec{a}\times b)\times z| = \sqrt{27}\left(6+\sqrt{6}\right)\dfrac{\sqrt{3}}{2}$

$\dfrac{9}{2}\left(6+\sqrt{6}\right)$

12. If all the words with or without meaning made using all the letters of the word "NAGPUR" are arranged as in a dictionary, then the word at 315^{th} position in this arrangement is :

(1) NRAGUP (2) NRAGPU

(3) NRAPGU (4) NRAPUG

Ans. (3)

Sol. NAGPUR

A → 5! = 120	
G ® 5! = 120	240
NA ® 4! = 24	264
NG ® 4! = 24	288
NP ® 4! = 24	312
NRAGPU = 1	313
NRAGUP	314
NRAPGU	315

13. Suppose for a differentiable function h, h(0) = 0, h(1) = 1 and h'(0) = h'(1) = 2. If $g(x) = h(e^x)\, e^{h(x)}$, then g'(0) is equal to:

(1) 5 (2) 3

(3) 8 (4) 4

Ans. (4)

Sol. $g(x) = h\left(e^x\right)\cdot e^{h(x)}$

$g'(x) = h(e^x)\cdot e^{h(x)}\cdot h'(x) + e^{h(x)}h'\left(e^x\right)\cdot e^x$

$g'(0) = h(1)e^{h(0)}h'(0) + e^{h(0)}h'(1)$

$= 2 + 2 = 4$

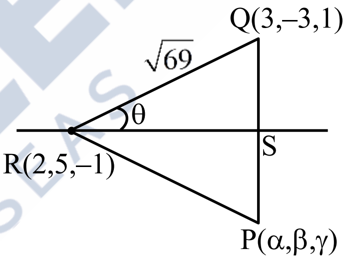
14. Let P (α, β, γ) be the image of the point Q(3, –3, 1) in the line $\dfrac{x-0}{1} = \dfrac{y-3}{1} = \dfrac{z-1}{-1}$ and R be the point (2, 5, –1). If the area of the triangle PQR is λ and $\lambda^2 = 14K$, then K is equal to:

(1) 36 (2) 72

(3) 18 (4) 81

Ans. (4)

Sol.



$RQ = \sqrt{1+64+4} = \sqrt{69}$

$\overrightarrow{RQ} = \hat{\ell} - 8\hat{j} + 2\hat{k}$

$\overrightarrow{RS} = \hat{\ell} + \hat{j} - \hat{k}$

$\cos\theta = \dfrac{\overrightarrow{RQ}\cdot\overrightarrow{RS}}{|\overrightarrow{RQ}||\overrightarrow{RS}|} = \left|\dfrac{1-8-2}{\sqrt{69}\sqrt{3}}\right| = \dfrac{9}{3\sqrt{23}}$

$\cos\theta = \dfrac{3}{\sqrt{23}} = \dfrac{RS}{RQ} = \dfrac{RS}{\sqrt{69}}$

$RS = 3\sqrt{3}$

$\sin\theta = \dfrac{\sqrt{14}}{\sqrt{23}} = \dfrac{QS}{\sqrt{69}}$

$QS = \sqrt{42}$

$\text{area} = \dfrac{1}{2}\cdot 2QS\cdot RS = \sqrt{42}\cdot 3\sqrt{3}$

$\lambda = 9\sqrt{14}$

$\lambda^2 = 81.14 = 14k$

$k = 81$

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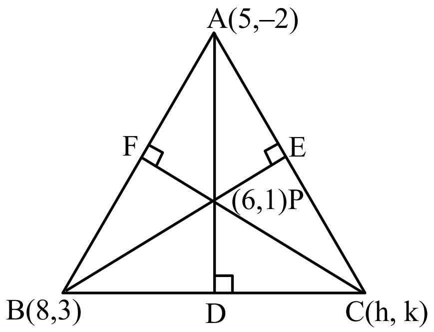
15. If P(6, 1) be the orthocentre of the triangle whose vertices are A(5, –2), B(8, 3) and C(h, k), then the point C lies on the circle.

(1) $x^2 + y^2 - 65 = 0$ (2) $x^2 + y^2 - 74 = 0$

(3) $x^2 + y^2 - 61 = 0$ (4) $x^2 + y^2 - 52 = 0$

Ans. (1)

Sol.



Slope of AD = 3

Slope of BC $= -\frac{1}{3}$

equation of BC = 3y + x – 17 = 0

slope of BE = 1

Slope of AC = –1

equation of AC is x + y – 3 = 0

point C is (–4, 7)

16. Let $f(x) = \frac{1}{7 - \sin 5x}$ be a function defined on R. Then the range of the function f(x) is equal to:

(1) $\left[\frac{1}{8}, \frac{1}{5}\right]$ (2) $\left[\frac{1}{7}, \frac{1}{6}\right]$

(3) $\left[\frac{1}{7}, \frac{1}{5}\right]$ (4) $\left[\frac{1}{8}, \frac{1}{6}\right]$

Ans. (4)

Sol. $\sin 5x \in [-1, 1]$

$-\sin 5x \in [-1, 1]$

$7 - \sin 5x \in [6, 8]$

$$\frac{1}{7 - \sin 5x} \in \left[\frac{1}{8}, \frac{1}{6}\right]$$

17. Let $\vec{a} = 2\hat{i} + \hat{j} - \hat{k}$, $\vec{b} = \left(\left(\vec{a} \times \left(\hat{i} + \hat{j}\right)\right) \times \hat{i}\right) \times \hat{i}$.

Then the square of the projection of $\vec{a}$ on $\vec{b}$ is :

(1) $\frac{1}{5}$ (2) 2

(3) $\frac{1}{3}$ (4) $\frac{2}{3}$

Ans. (2)

Sol. $\vec{a} \times \left(\hat{i} + \hat{j}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ 1 & 1 & 0 \end{vmatrix}$

$= \hat{i} - \hat{j} + \hat{k}$

$\left(\vec{a} \times \left(\hat{i} \times \hat{j}\right)\right) \times \hat{i} = \hat{k} + \hat{j}$

$\left(\left(\vec{a} \times \left(\hat{i} \times \hat{j}\right)\right) \times i\right) \times \hat{i} = \hat{j} - \hat{k}$

projection of $\vec{a}$ on $\hat{b}$ $= \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$

$= \frac{1+1}{\sqrt{2}} = \sqrt{2}$

18. If the area of the region

$\left\{(x, y) : \frac{a}{x^2} \le y \le \frac{1}{x}, 1 \le x \le 2, 0 < a < 1\right\}$ is

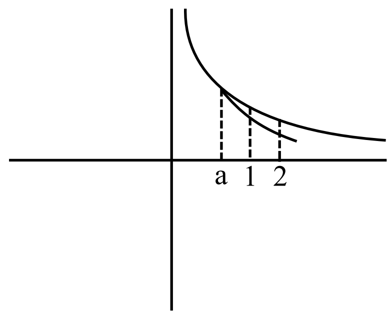
$(\log_e 2) - \frac{1}{7}$ then the value of 7a – 3 is equal to:

(1) 2 (2) 0

(3) –1 (4) 1

Ans. (3)

Sol.



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area $\int_1^2 \left(\frac{1}{x} - \frac{a}{x^2}\right)dx$

$\left[\ell nx + \frac{a}{x}\right]_1^2$

$\ell n2 + \frac{a}{2} - a = \log_e 2 - \frac{1}{7}$

$\frac{-a}{2} = -\frac{1}{7}$

$a = \frac{2}{7}$

$7a = 2$

$7a - 3 = -1$

19. If $\int \frac{1}{a^2 \sin^2 x + b^2 \cos^2 x} dx = \frac{1}{12} \tan^{-1}(3\tan x) +$ constant, then the maximum value of $a\sin x + b\cos x$, is :

(1) $\sqrt{40}$ (2) $\sqrt{39}$

(3) $\sqrt{42}$ (4) $\sqrt{41}$

Ans. (1)

Sol. $\int \frac{\sec^2 x dx}{a^2 \tan^2 x + b^2}$

let $\tan x = t$

$\sec^2 x dx = dt$

$\int \frac{dt}{a^2 t^2 + b^2}$

$\frac{1}{a^2} \int \frac{dt}{t^2 + \left(\frac{b}{a}\right)^2}$

$\frac{1}{a^2} \frac{1}{\frac{b}{a}} \tan^{-1}\left(\frac{t}{b} a\right) + c$

$\frac{1}{ab} \tan^{-1}\left(\frac{\alpha}{b} \tan x\right) + c$

on comparing $\frac{a}{b} = 3$

$ab = 12$

$a = 6, b = 2$

maximum value of

$6\sin x + 2\cos x$ is $\sqrt{40}$

20. If A is a square matrix of order 3 such that

$\det(A) = 3$ and

$\det(\operatorname{adj}(-4 \operatorname{adj}(-3 \operatorname{adj}(3 \operatorname{adj}((2A)^{-1}))))) = 2^m 3^n$,

then m +| 2n is equal to:

(1) 3 (2) 2

(3) 4 (4) 6

Ans. (3)

Sol. $|A| = 3$

$\left|\operatorname{adj}(-4\operatorname{adj}(-3\operatorname{adj}(3\operatorname{adj}\left((2A)^{-1}\right))))\right|$

$\left|-4\operatorname{adj}\left(-3\operatorname{adj}(3\operatorname{adj}(2A)^{-1}\right)\right|^2$

$4^6 \left|\operatorname{adj}\left(-3\operatorname{adj}\left(3\operatorname{adj}(2A)^{-1}\right)\right)\right|^2$

$2^{12} \cdot 3^{12} \left|3\operatorname{adj}(2A)^{-1}\right|^8$

$2^{12} \cdot 3^{12} \cdot 3^{24} \left|\operatorname{adj}(2A)^{-1}\right|^8$

$2^{12} \cdot 3^{36} \left|(2A)^{-1}\right|^{16}$

$2^{12} \cdot 3^{36} \frac{1}{|2A|^{16}}$

$2^{12} \cdot 3^{36} \frac{1}{2^{48} |A|^{16}}$

$2^{12} \cdot 3^{36} \frac{1}{2^{48} \cdot 3^{16}}$

$\frac{3^{20}}{2^{36}} = 2^{-36} \cdot 3^{20}$

$m = -36 \quad n = 20$

$m + 2n = 4$

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SECTION-B

21. Let [t] denote the greatest integer less than or equal to t. Let f: $[0, \infty) \to R$ be a function defined by $f(x)=\left[\frac{x}{2}+3\right]-\left[\sqrt{x}\right]$. Let S be the set of all points in the interval [0, 8] at which f is not continuous. Then $\sum_{a\in S} a$ is equal to ________.

Ans. (17)

Sol. $\left[\frac{x}{2}+3\right]$ is discontinuous at x = 2,4,6,8

$\sqrt{x}$ is discontinuous at x = 1,4

F(x) is discontinuous at x = 1,2,6,8

$\sum a = 1+2+6+8=17$

22. The length of the latus rectum and directrices of a hyperbola with eccentricity e are 9 and $x = \pm\frac{4}{\sqrt{3}}$, respectively. Let the line $y - \sqrt{3}x + \sqrt{3} = 0$ touch this hyperbola at (x_0, y_0). If m is the product of the focal distances of the point (x_0, y_0), then $4e^2 + m$ is equal to ________.

NTA Ans. (61)

Ans. (Bonus)

Sol. Given $\frac{2b^2}{a} = 9$ and $\frac{a}{e} = \pm\frac{4}{\sqrt{3}}$

equation of tangent $y - \sqrt{3}x + \sqrt{3} = 0$

by equation of tangent

Let slope = S = $\sqrt{3}$

Constant = $-\sqrt{3}$

By condition of tangency

$\Rightarrow 6 = 6a^2 - 9a$

$\Rightarrow a = 2, b^2 = 9$

Equation of Hyperbola is

$\frac{x^2}{4} - \frac{y^2}{9} = 1$ and for tangent

Point of contact is $(4, 3\sqrt{3}) = (x_0, y_0)$

Now $e = \sqrt{1+\frac{9}{4}} = \frac{\sqrt{13}}{2}$

Again product of focal distances

$m = (x_0e + a)(x_0e - a)$

$m + 4e^2 = 20e^2 - a^2$

$= 20 \times \frac{13}{4} - 4 = 61$

(There is a printing mistake in the equation of directrix $x = \pm\frac{4}{\sqrt{3}}$.

Corrected equation is $x = \pm\frac{4}{\sqrt{13}}$ for directrix, as eccentricity must be greater than one, so question must be bonus)

23. If $S(x) = (1+x) + 2(1+x)^2 + 3(1+x)^3 + \ldots + 60(1+x)^{60}$, $x \neq 0$, and $(60)^2 S(60) = a(b)^b + b$, where a, b $\in$ N, then (a + b) equal to ______

Ans. (3660)

Sol.

$S(x)=(1+x)+2(1+x)^2+3(1+x)^3+\ldots+60(1+x)^{60}$

$(1+x)S = \quad (1+x)^2 + \ldots\ldots \quad 59(1+x)^{60} + 60(1+x)^{61}$

$$-xS = \frac{(1+x)\left((1+x)^{60}-1\right)}{x} - 60(1+x)^{61}$$

Put x = 60

$$-60S = \frac{61\left((61)^{60}-1\right)}{60} - 60(61)^{61}$$

on solving 3660

24. Let [t] denote the largest integer less than or equal to t. If

$$\int_0^3\left([x^2]+\left[\frac{x^2}{2}\right]\right)dx = a + b\sqrt{2} - \sqrt{3} - \sqrt{5} + c\sqrt{6} - \sqrt{7},$$

where a, b, c $\in$ z, then a + b + c is equal to ______

Ans. (23)

Sol. $\int_0^3 [x^2]dx + \int_0^3\left[\frac{x^2}{2}\right]dx$

$= \int_0^1 0\,dx + \int_1^{12} 1\,dx + \int_{\sqrt{2}}^{\sqrt{3}} 2\,dx$

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$$+\int_{\sqrt{3}}^{2} 3\,dx+\int_{2}^{\sqrt{5}} 4\,dx+\int_{\sqrt{5}}^{\sqrt{6}} 5\,dx$$

$$+\int_{\sqrt{6}}^{\sqrt{7}} 6\,dx+\int_{\sqrt{7}}^{\sqrt{8}} 7\,dx+\int_{\sqrt{8}}^{3} 8\,dx$$

$$+\int_{0}^{\sqrt{2}} 0\,dx+\int_{\sqrt{2}}^{2} 1\,dx$$

$$+\int_{2}^{\sqrt{6}} 2\,dx+\int_{\sqrt{6}}^{\sqrt{8}} 3\,dx+\int_{\sqrt{8}}^{3} 4dx=31-6\sqrt{2}-\sqrt{3}-\sqrt{5}$$

$$-2\sqrt{6}-\sqrt{7}$$

$a=31\quad b=-6\quad c=-2$

$a+b+c=31-6-2=23$

25. From a lot of 12 items containing 3 defectives, a sample of 5 items is drawn at random. Let the random variable X denote the number of defective items in the sample. Let items in the sample be drawn one by one without replacement. If variance of X is $\frac{m}{n}$, where gcd(m, n) = 1, then n – m is equal to _______.

Ans. (71)

Sol. $a=1-\frac{{}^{3}C_5}{{}^{12}C_5}$

$b=3.\frac{{}^{9}C_4}{{}^{12}C_5}$

$c=3.\frac{{}^{9}C_3}{{}^{12}C_5}$

$d=1.\frac{{}^{9}C_2}{{}^{12}C_5}$

$u = 0.a + 1.b + 2.c + 3.d = 1.25$

$\sigma^2 = 0.a + 1.b + 4.c + 9d - u^2$

$\sigma^2=\frac{105}{176}$

Ans. 176 – 105 = 71

26. In a triangle ABC, BC = 7, AC = 8, AB = $\alpha \in N$ and $\cos A = \frac{2}{3}$. If $49\cos(3C) + 42 = \frac{m}{n}$, where gcd(m, n) = 1, then m + n is equal to _______

Ans. (39)

26. In a triangle ABC, BC = 7, AC = 8, AB = $\alpha \in N$ and $\cos A = \frac{2}{3}$. If $49\cos(3C) + 42 = \frac{m}{n}$, where gcd(m, n) = 1, then m + n is equal to _______

Ans. (39)

Sol. $\cos A=\frac{b^2+c^2-a^2}{2bc}$

$\frac{2}{3}=\frac{8^2+c^2-7^2}{2\times 8\times c}$

$C = 9$

$\cos C=\frac{7^2+8^2-9^2}{2\times 7\times 8}=\frac{2}{7}$

$49\cos 3C + 42$

$49(4\cos^3 C - 3\cos C) + 42$

$49\left(4\left(\frac{2}{7}\right)^3-3\left(\frac{2}{7}\right)\right)+42$

$=\frac{32}{7}$

$m + n = 32 + 7 = 39$

27. If the shortest distance between the lines $\frac{x-\lambda}{3}=\frac{y-2}{-1}=\frac{z-1}{1}$ and $\frac{x+2}{-3}=\frac{y+5}{2}=\frac{z-4}{4}$ is $\frac{44}{\sqrt{30}}$, then the largest possible value of $|\lambda|$ is equal to _______.

Ans. (43)

Sol. $\bar{a}_1=\lambda\hat{i}+2\hat{j}+\hat{k}$

$\bar{a}_2=-2\hat{i}-5\hat{j}+4\hat{k}$

$\vec{p}-=3\hat{i}-\hat{j}+\hat{k}$

$\vec{q}-=-3\hat{i}+2\hat{j}+4\hat{k}$

$(\lambda+2)\hat{i}+7\hat{j}-3\hat{k}=\bar{a}_1-\bar{a}_2$

$\vec{p}\times\vec{q}-=-6\hat{i}-15\hat{j}+3\hat{k}$

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$$\frac{44}{\sqrt{30}}=\frac{|-6\lambda-12-105-9|}{\sqrt{(-6)^2+(-15)^2+3^2}}$$

$$\frac{44}{\sqrt{30}}=\frac{|6\lambda+126|}{3\sqrt{30}}$$

$$132=|6\lambda+126|$$

$\lambda = 1, \lambda = -43$

$|\lambda| = 43$

28. Let α, β be roots of $x^2+\sqrt{2}x-8=0$.

If $U_n=\alpha^n+\beta^n$, then $\frac{U_{10}+\sqrt{12}U_9}{2U_8}$

is equal to _______.

Ans. (4)

Sol. $$\frac{\alpha^{10}+\beta^{10}+\sqrt{2}\left(\alpha^9+\beta^9\right)}{2\left(\alpha^8+\beta^8\right)}$$

$$\frac{\alpha^8\left(\alpha^2+\sqrt{2}\alpha\right)+\beta^8\left(\beta^2+\sqrt{2}\beta\right)}{2\left(\alpha^8+\beta^8\right)}$$

$$\frac{8\alpha^8+8\beta^8}{2\left(\alpha^8+\beta^8\right)}=4$$

29. If the system of equations

$2x + 7y + \lambda z = 3$

$3x + 2y + 5z = 4$

$x + \mu y + 32z = -1$

has infinitely many solutions, then $(\lambda - \mu)$ is equal to ________ :

Ans. (38)

Sol. $D = D_1 = D_2 = D_3 = 0$

$$D_3=\begin{vmatrix}2 & 7 & 3\\ 3 & 2 & 4\\ 1 & \mu & -1\end{vmatrix}=0\Rightarrow\mu=-39$$

$$D=\begin{vmatrix}2 & 7 & \lambda\\ 3 & 2 & 5\\ 1 & -39 & 32\end{vmatrix}=0\Rightarrow\lambda=-1$$

$$\lambda-\mu=38$$

30. If the solution $y(x)$ of the given differential equation $(e^y + 1)\cos x\,dx + e^y \sin x\,dy = 0$ passes through the point $\left(\frac{\pi}{2},0\right)$, then the value of $e^{y\left(\frac{\pi}{6}\right)}$ is equal to ______.

Ans. (3)

Sol. $(e^y + 1)\cos x\,dx + e^y \sin x\,dy = 0$

$$\Rightarrow d\left(\left(e^y+1\right)\sin x\right)=0$$

$$\left(e^y+1\right)\sin x=C$$

It passes through $\left(\frac{\pi}{2},0\right)$

$$\Rightarrow c=2$$

Now, $x=\frac{\pi}{6}$

$$\Rightarrow e^y=3$$

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