FINAL JEE–MAIN EXAMINATION – APRIL, 2024

(Held On Monday 08th April, 2024) **TIME : 9 : 00 AM to 12 : 00 NOON**

MATHEMATICS — TEST PAPER WITH SOLUTION

SECTION-A

1. The value of k $\in \mathbb{N}$ for which the integral $I_n = \int_0^1 (1-x^k)^n dx,\ n \in \mathbb{N}$, satisfies 147 I_{20} = 148 I_{21} is :

(1) 10 (2) 8

(3) 14 (4) 7

Ans. (4)

Sol. $I_n = \int_0^1 (1-x^k)^n.1\,dx$

$I_n = (1-x^k)^n.x - nk\int_0^1 (1-x^k)^{n-1}.x^{k-1}.dx$

$I_n = nk\int_0^1 [(1-x^k)^n - (1-x^k)^{n-1}]dx$

$I_n = nkI_n - nkI_n$

$\frac{I_n}{I_{n-1}} = \frac{nk}{nk+1}$

$\frac{I_{21}}{I_{20}} = \frac{21k}{1+21k}$

$= \frac{147}{148} \Rightarrow k = 7$

2. The sum of all the solutions of the equation $(8)^{2x} - 16\cdot(8)^x + 48 = 0$ is :

(1) $1 + \log_6(8)$ (2) $\log_8(6)$

(3) $1 + \log_8(6)$ (4) $\log_8(4)$

Ans. (3)

Sol. $(8)^{2x} - 16\cdot(8)^x + 48 = 0$

Put $8^x = t$

$t^2 - 16 + 48 = 0$

$\Rightarrow t = 4$ or $t = 12$

$\Rightarrow 8^x = 4 \quad 8^x = 12$

$\Rightarrow x = \log_8 x \quad x = \log_8 12$

sum of solution = $\log_8 4 + \log_8 12$

$= \log_8 48 = \log_8(6.8)$

$= 1 + \log_8 6$

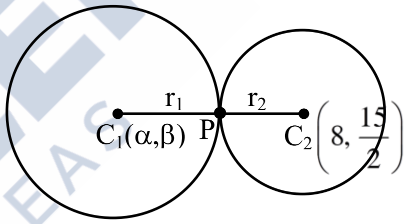
3. Let the circles $C_1 : (x-\alpha)^2 + (y-\beta)^2 = r_1^2$ and $C_2 : (x-8)^2 + \left(y-\frac{15}{2}\right)^2 = r_2^2$ touch each other externally at the point (6, 6). If the point (6, 6) divides the line segment joining the centres of the circles C_1 and C_2 internally in the ratio 2 : 1, then $(\alpha+\beta) + 4\left(r_1^2 + r_2^2\right)$ equals

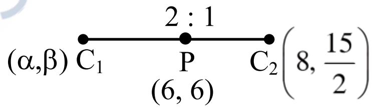
(1) 110 (2) 130

(3) 125 (4) 145

Ans. (2)

Sol.





$\because \frac{16+\alpha}{3} = 6$ and $\frac{15+\beta}{3} = 6$

$\Rightarrow (\alpha, \beta) \equiv (2, 3)$

Also, $C_1C_2 = r_1 + r_2$

$\Rightarrow \sqrt{(2-8)^2 + \left(3-\frac{15}{2}\right)^2} = 2r_2 + r_2$

$\Rightarrow r_2 = \frac{5}{2} \Rightarrow r_1 = 2r_2 = 5$

$\therefore (\alpha+\beta) + 4(r_1^2 + r_2^2)$

$= 5 + 4\left(\frac{25}{4} + 25\right) = 130$

4. Let P(x, y, z) be a point in the first octant, whose projection in the xy-plane is the point Q. Let OP = γ ; the angle between OQ and the positive x-axis be θ; and the angle between OP and the positive z-axis be ϕ, where O is the origin. Then the distance of P from the x-axis is :

(1) $\gamma\sqrt{1-\sin^2\phi\cos^2\theta}$ (2) $\gamma\sqrt{1+\cos^2\theta\sin^2\phi}$

(3) $\gamma\sqrt{1-\sin^2\theta\cos^2\phi}$ (4) $\gamma\sqrt{1+\cos^2\phi\sin^2\theta}$

Ans. (1)

Sol. P(x, y, z), Q(x, y, O) ; $x^2+y^2+z^2=\gamma^2$

$\overline{OQ} = x\hat{i}+y\hat{j}$

$\cos\theta = \dfrac{x}{\sqrt{x^2+y^2}}$

$\cos\phi = \dfrac{z}{\sqrt{x^2+y^2+z^2}}$

$\Rightarrow \sin^2\phi = \dfrac{x^2+y^2}{x^2+y^2+z^2}$

distance of P from x-axis $\sqrt{y^2+z^2}$

$\Rightarrow \sqrt{\gamma^2-x^2} \Rightarrow \gamma\sqrt{1-\dfrac{x^2}{\gamma^2}}$

$= \gamma\sqrt{1-\cos^2\theta\sin^2\phi}$

5. The number of critical points of the function $f(x) = (x-2)^{2/3}(2x+1)$ is :

(1) 2 (2) 0

(3) 1 (4) 3

Ans. (1)

Sol. $f(x) = (x-2)^{2/3}(2x+1)$

$f'(x) = \dfrac{2}{3}(x-2)^{-1/3}(2x+1) + (x-2)^{2/3}(2)$

$f'(x) = 2 \times \dfrac{(2x+1)+(x-2)}{3(x-2)^{1/3}}$

$\dfrac{3x-1}{(x-2)^{1/3}} = 0$

Critical points $x = \dfrac{1}{3}$ and x = 2

6. Let f(x) be a positive function such that the area bounded by y = f(x), y = 0 from x = 0 to x = a > 0 is $e^{-a}+4a^2+a-1$. Then the differential equation, whose general solution is $y = c_1f(x)+c_2$, where c_1 and c_2 are arbitrary constants, is :

(1) $(8e^x-1)\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}=0$

(2) $(8e^x+1)\dfrac{d^2y}{dx^2}-\dfrac{dy}{dx}=0$

(3) $(8e^x+1)\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}=0$

(4) $(8e^x-1)\dfrac{d^2y}{dx^2}-\dfrac{dy}{dx}=0$

Ans. (3)

Sol. $\int_0^a f(x)dx = e^{-a}+4a^2+a-1$

$f(a) = -e^{-a}+8a+1$

$f(x) = -e^{-x}+8x+1$

Now $y = C_1f(x)+C_2$

$\dfrac{dy}{dx} = C_1f'(x) = C_1(e^{-x}+8)$ .....(1)

$\dfrac{d^2y}{dx^2} = -C_1e^{-x} \Rightarrow -e^x\dfrac{d^2y}{dx^2}$

Put in equation (1)

$\dfrac{dy}{dx} = -e^x\dfrac{d^2y}{dx^2}(e^{-x}+8)$

$(8e^x+1)\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}=0$

7. Let $f(x) = 4\cos^3 x + 3\sqrt{3}\cos^2 x - 10$. The number of points of local maxima of f in interval $(0, 2\pi)$ is:

(1) 1 (2) 2

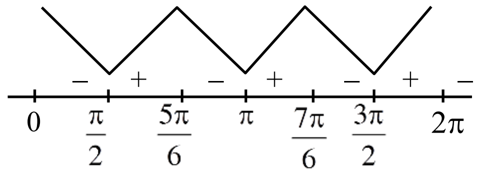
(3) 3 (4) 4

Ans. (2)

Sol. $f(x) = 4\cos^3(x) + 3\sqrt{3}\cos^2(x) - 10$; $x \in (0, 2\pi)$

$\Rightarrow f'(x) = 12\cos^2 x[-\sin(x)] + 3\sqrt{3}(2\cos(x))[-\sin(x)]$

$\Rightarrow f'(x) = -6\sin(x)\cos(x)[2\cos(x) + \sqrt{3}]$



local maxima at $x = \frac{5\pi}{6}, \frac{7\pi}{6}$

8. Let $A = \begin{bmatrix} 2 & a & 0 \\ 1 & 3 & 1 \\ 0 & 5 & b \end{bmatrix}$. If $A^3 = 4A^2 - A - 21I$, where I is the identity matrix of order 3 × 3, then 2a + 3b is equal to :

(1) –10 (2) –13

(3) –9 (4) –12

Ans. (2)

Sol. $A^3 - 4A^2 + A + 21I = 0$

$tr(A) = 4 = 5 + 6 \Rightarrow b = -1$

$|A| = -21$

$-16 + a = -21 \Rightarrow a = -5$

$2a + 3b = -13$

9. If the shortest distance between the lines

$L_1 : \vec{r} = (2+\lambda)\hat{i} + (1-3\lambda)\hat{j} + (3+4\lambda)\hat{k}, \lambda \in \mathbb{R}$

$L_2 : \vec{r} = 2(1+\mu)\hat{i} + 3(1+\mu)\hat{j} + (5+\mu)\hat{k}, \mu \in \mathbb{R}$

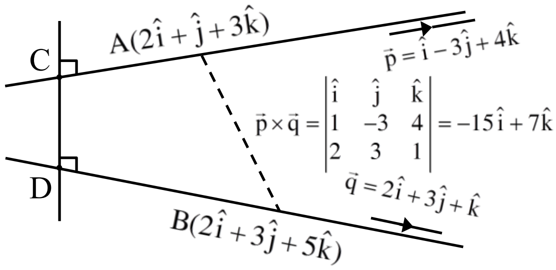
is $\frac{m}{\sqrt{n}}$, where gcd (m, n) = 1, then the value of m + n equals.

(1) 384 (2) 387

(3) 377 (4) 390

Ans. (2)

Sol.



Shortes distance (CD) = $\left|\frac{\overline{AB}\cdot\vec{p}\times\vec{q}}{|\vec{p}\times\vec{q}|}\right|$

$= \left|\frac{(0\hat{i}+2\hat{j}+2\hat{k}).(-15\hat{i}+7\hat{j}+9\hat{k})}{\sqrt{355}}\right|$

$= \frac{0+14+18}{\sqrt{355}} = \frac{32}{\sqrt{355}}$

$\therefore m + n = 32 + 355 = 387$

10. Let the sum of two positive integers be 24. If the probability, that their product is not less than $\frac{3}{4}$ times their greatest positive product, is $\frac{m}{n}$, where gcd(m, n) = 1, then n – m equals :

(1) 9 (2) 11

(3) 8 (4) 10

Ans. (4)

Sol. $x + y = 24$, $x, y \in N$

$AM > GM \Rightarrow xy \le 144$

$xy \ge 108$

Favorable pairs of (x, y) are

(13, 11), (12, 12), (14, 10), (15, 9), (16, 8), (17, 7), (18, 6), (6, 18), (7, 17), (8, 16), (9, 15), (10, 14), (11, 13)

i.e. 13 cases

Total choices for x + y = 24 is 23

Probability = $\frac{13}{23} = \frac{m}{n}$

$n - m = 10$

11. If $\sin x = -\frac{3}{5}$, where $\pi < x < \frac{3\pi}{2}$, then $80(\tan^2 x - \cos x)$ is equal to :

(1) 109 (2) 108

(3) 18 (4) 19

Ans. (1)

Sol. $\sin x = \frac{-3}{5}$, $\pi < x < \frac{3\pi}{2}$

$\tan x = \frac{3}{4}$ $\cos x = -\frac{4}{5}$

$80(\tan^2 x - \cos x)$

$= 80\left(\frac{9}{16} + \frac{4}{5}\right) = 45 + 64 = 109$

12. Let $I(x) = \int \frac{6}{\sin^2 x(1-\cot x)^2} dx$. If $I(0) = 3$, then $I\left(\frac{\pi}{12}\right)$ is equal to :

(1) $\sqrt{3}$ (2) $3\sqrt{3}$

(3) $6\sqrt{3}$ (4) $2\sqrt{3}$

Ans. (2)

Sol. $I(x) = \int \frac{6dx}{\sin^2 x(1-\cot x)^2} = \int \frac{6\operatorname{cosec}^2 x\, dx}{(1-\cot x)^2}$

Put $1 - \cot x = t$

$\operatorname{cosec}^2 x\, dx = dt$

$I = \int \frac{6dt}{t^2} = \frac{-6}{t} + c$

$I(x) = \frac{-6}{1-\cot x} c$, $c = 3$

$I(x) = 3 - \frac{6}{1-\cot x}$, $I\left(\frac{\pi}{12}\right) = 3 - \frac{6}{1-(2+\sqrt{3})}$

$I\left(\frac{\pi}{12}\right) = 3 + \frac{6}{\sqrt{3}+1} = 3 + \frac{6(\sqrt{3}-1)}{2} = 3\sqrt{3}\sqrt{2}$

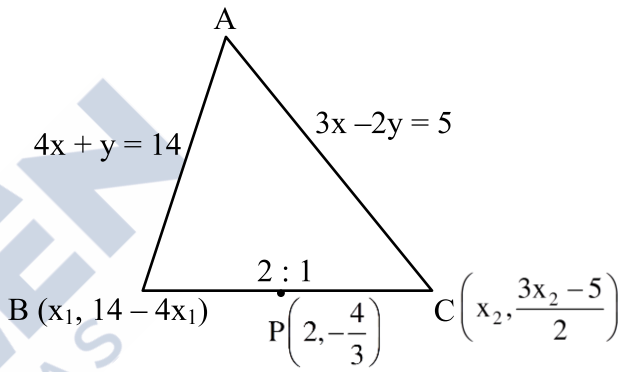
13. The equations of two sides AB and AC of a triangle ABC are $4x + y = 14$ and $3x - 2y = 5$, respectively. The point $\left(2, -\frac{4}{3}\right)$ divides the third side BC internally in the ratio 2 : 1. The equation of the side BC is :

(1) $x - 6y - 10 = 0$ (2) $x - 3y - 6 = 0$

(3) $x + 3y + 2 = 0$ (4) $x + 6y + 6 = 0$

Ans. (3)

Sol.



$\frac{2x_2 + x_1}{3} = 2$, $\frac{2\left(\frac{3x_2-5}{2}\right) + (14-4x_1)}{3} = \frac{-4}{3}$

$2x_2 + x_1 = 6$, $3x_2 - 4x_1 = -13$

$x_2 = 1$, $x_1 = 4$

So, $C(1, -1)$, $B(4, -2)$

$m = \frac{-1}{3}$

Equation of BC : $y + 1 = \frac{-1}{3}(x - 1)$

$3y + 3 = -x + 1$

$x + 3y + 2 = 0$

14. Let [t] be the greatest integer less than or equal to t. Let A be the set of al prime factors of 2310 and $f: A \to \mathbb{Z}$ be the function $f(x) = \left[\log_2\left(x^2 + \left[\frac{x^3}{5}\right]\right)\right]$.

The number of one-to-one functions from A to the range of f is :

(1) 20 (2) 120

(3) 25 (4) 24

Ans. (2)

Sol. N = 2310 = 231 × 10

= 3 × 11 × 7 × 2 × 5

A = {2, 3, 5, 7, 11}

$f(x) = \left[\log_2\left(x^2 + \left[\frac{x^3}{5}\right]\right)\right]$

$f(2) = [\log_2(5)] = 2$

$f(3) = [\log_2(14)] = 3$

$f(5) = [\log_2(25 + 25)] = 5$

$f(7) = [\log_2(117)] = 6$

$f(11) = [\log_2 387] = 8$

Range of f : B = {2, 3, 5, 6, 8}

No. of one-one functions = 5! = 120

15. Let z be a complex number such that |z + 2| = 1 and $\text{Im}\left(\frac{z+1}{z+2}\right) = \frac{1}{5}$. Then the value of $\left|\text{Re}\left(\overline{z+2}\right)\right|$ is :

(1) $\frac{\sqrt{6}}{5}$ (2) $\frac{1+\sqrt{6}}{5}$

(3) $\frac{24}{5}$ (4) $\frac{2\sqrt{6}}{5}$

Ans. (4)

Sol. $|z + 2| = 1,\ \text{Im}\left(\frac{z+1}{z+2}\right) = \frac{1}{5}$

Let $z + 2 = \cos\theta + i\sin\theta$

$\frac{1}{z+2} = \cos\theta - i\sin\theta$

$\Rightarrow \frac{z+1}{z+2} = 1 - \frac{1}{z+2} = 1 - (\cos\theta - i\sin\theta)$

$= (1 - \cos\theta) + i\sin\theta$

$\text{Im}\left(\frac{z+1}{z+2}\right) = \sin\theta,\ \sin\theta = \frac{1}{5}$

$\cos\theta = \pm\sqrt{1 - \frac{1}{25}} = \pm\frac{2\sqrt{6}}{5}$

$\left|\text{Re}(\overline{z+2})\right| = \frac{2\sqrt{6}}{5}$

16. If the set R = {(a, b) ; a + 5b = 42, a, b ∈ ℕ} has m elements and $\sum_{n=1}^{m}\left(1 + i^{n!}\right) = x + iy$, where $I = \sqrt{-1}$, then the value of m + x + y is :

(1) 8 (2) 12

(3) 4 (4) 5

Ans. (2)

Sol. a + 5b = 42, a, b ∈ N

a = 42 – 5b, b = 1, a = 37

b = 2, a = 32

b = 3, a = 27

⋮

b = 8, a = 2

R has "8" elements ⇒ m = 8

$\sum_{n=1}^{8}(1 - i^{n!}) = x + iy$

for n ≥ 4, $i^{n!} = 1$

$\Rightarrow (1 - i) + (1 - i^{2!}) + (1 - i^{3!})$

= 1 – I + 2 + 1 + 1

= 5 – I = x + iy

m + x + y = 8 + 5 – 1 = 12

17. For the function $f(x) = (\cos x) - x + 1, x \in \mathbb{R}$, between the following two statements

(S1) $f(x) = 0$ for only one value of x is $[0, \pi]$.

(S2) $f(x)$ is decreasing in $\left[0, \frac{\pi}{2}\right]$ and increasing in $\left[\frac{\pi}{2}, \pi\right]$.

(1) Both (S1) and (S2) are correct
(2) Only (S1) is correct
(3) Both (S1) and (S2) are incorrect
(4) Only (S2) is correct

Ans. (2)

Sol. $f(x) = \cos x - x + 1$

$f'(x) = -\sin x - 1$

f is decreasing $\forall x \in R$

$f(x) = 0$

$f(0) = 2, f(\pi) = -\pi$

f is strictly decreasing in $[0, \pi]$ and $f(0).f(\pi) < 0$

$\Rightarrow$ only one solution of $f(x) = 0$

S1 is correct and S2 is incorrect.

18. The set of all α, for which the vector $\vec{a} = \alpha t\hat{i} + 6\hat{j} - 3\hat{k}$ and $\vec{b} = t\hat{i} - 2\hat{j} - 2\alpha t\hat{k}$ are inclined at an obtuse angle for all $t \in \mathbb{R}$ is :

(1) $[0, 1)$
(2) $(-2, 0]$
(3) $\left(-\frac{4}{3}, 0\right]$
(4) $\left(-\frac{4}{3}, 1\right)$

Ans. (3)

Sol. $\vec{a} = \alpha t\hat{i} + 6\hat{j} - 3\hat{k}$

$\vec{b} = t\hat{i} - 2\hat{j} - 2\alpha t\hat{k}$

so $\vec{a}.\vec{b} < 0, \ \forall t \in R$

$\alpha t^2 - 12 + 6\alpha t < 0$

$\alpha t^2 + 6\alpha t - 12 < 0, \ \forall t \in R$

$\alpha < 0$, and $D < 0$

$36\alpha^2 + 48\alpha < 0$

$12\alpha(3\alpha + 4) < 0$

$\frac{-4}{3} < \alpha < 0$

also for $a = 0, \ \vec{a}.\vec{b} < 0$

hence a $\alpha \in \left(\frac{-4}{3}, 0\right]$

19. Let $y = y(x)$ be the solution of the differential equation $(1 + y^2)e^{\tan x}dx + \cos^2 x(1 + e^{2\tan x})dy = 0$, $y(0) = 1$. Then $y\left(\frac{\pi}{4}\right)$ is equal to :

(1) $\frac{2}{e}$
(2) $\frac{1}{e^2}$
(3) $\frac{1}{e}$
(4) $\frac{2}{e^2}$

Ans. (3)

Sol. $(1 + y^2)\,e^{\tan x}dx + \cos^2 x(1 + e^{2\tan x})\,dy = 0$

$$\int \frac{\sec^2 x\, e^{\tan x}}{1 + e^{2\tan x}}dx + \int \frac{dy}{1 + y^2} = C$$

$\Rightarrow \tan^{-1}(e^{\tan x}) + \tan^{-1} y = C$

for $x = 0, y = 1, \tan^{-1}(1) + \tan^{-1} 1 = C$

$C = \frac{\pi}{2}$

$\tan^{-1}(e^{\tan x}) + \tan^{-1} y = \frac{\pi}{2}$

Put $x = \pi, \tan^{-1} e + \tan^{-1} y = \frac{\pi}{2}$

$\tan^{-1} y = \cot^{-1} e$

$y = \frac{1}{e}$

20. Let $H : \frac{-x^2}{a^2} + \frac{y^2}{b^2} = 1$ be the hyperbola, whose eccentricity is $\sqrt{3}$ and the length of the latus rectum is $4\sqrt{3}$. Suppose the point $(\alpha, 6)$, $\alpha > 0$ lies on H. If β is the product of the focal distances of the point $(\alpha, 6)$, then $\alpha^2 + \beta$ is equal to :

(1) 170
(2) 171
(3) 169
(4) 172

Ans. (2)

Sol. $H: \frac{y^2}{b^2} - \frac{x^2}{a^2} = 1,\ e = \sqrt{3}$

$$e = \sqrt{1 + \frac{a^2}{b^2}} = \sqrt{3} \quad \Rightarrow \frac{a^2}{b^2} = 2$$

$$a^2 = 2b^2$$

$$\text{length of L.R.} = \frac{2a^2}{b} = 4\sqrt{3}$$

$$a = \sqrt{6}$$

$$P(\alpha, 6) \text{ lie on } \frac{y^2}{3} - \frac{x^2}{6} = 1$$

$$12 - \frac{\alpha^2}{6} = 1 \Rightarrow \alpha^2 = 66$$

Foci = (0, ±be) = (0, 3) & (0, –3)

Let d_1 & d_2 be focal distances of P(α, 6)

$$d_1 = \sqrt{\alpha^2 + (6+be)^2},\ d_2 = \sqrt{\alpha^2 + (6-be)^2}$$

$$d_1 = \sqrt{66+81},\ d_2 = \sqrt{66+9}$$

$$\beta = d_1 d_2 = \sqrt{147 \times 75} = 105$$

$$\alpha^2 + \beta = 66 + 105 = 171$$

SECTION-B

21. Let $A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$. If the sum of the diagonal elements of A^{13} is 3^n, then n is equal to ______.

Ans. (7)

Sol. $A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ 3 & 0 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 3 & -3 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -6 \\ 6 & -3 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 3 & -6 \\ 6 & -3 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -9 \\ 9 & -9 \end{bmatrix}$$

$$A^5 = \begin{bmatrix} 0 & -9 \\ 9 & -9 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -9 & -9 \\ 9 & -18 \end{bmatrix}$$

$$A^6 = \begin{bmatrix} -9 & -9 \\ 9 & -18 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -27 & 0 \\ 0 & -27 \end{bmatrix}$$

$$A^7 = \begin{bmatrix} -27 & -0 \\ 0 & -27 \end{bmatrix}\begin{bmatrix} -54 & 27 \\ -27 & -27 \end{bmatrix} = \begin{bmatrix} 3^6 \times 2 & -27^2 \\ 27^2 & 3^6 \end{bmatrix}$$

$$3^7 = 3^n \Rightarrow n = 7$$

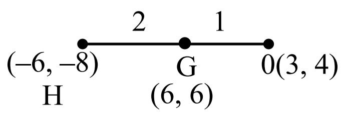
22. If the orthocentre of the triangle formed by the lines 2x + 3y – 1 = 0, x + 2y – 1 = 0 and ax + by – 1 = 0, is the centroid of another triangle, whose circumecentre and orthocentre respectively are (3, 4) and (–6, –8), then the value of |a – b| is _____.

Ans. (16)

Sol. 2x + 3y – 1 = 0

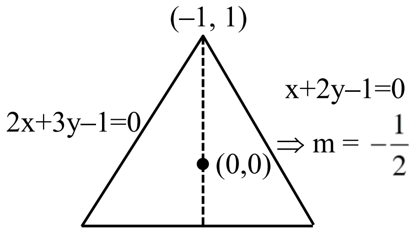
x + 2y – 1 = 0

ax + by – 1 = 0



$$\left(\frac{6-6}{3}, \frac{8-8}{3}\right)$$

= (0, 0)



$ax + by - 1 = 0$

$$\left(\frac{1-0}{-1-0}\right)\left(\frac{-a}{b}\right) = -1$$

$\Rightarrow -a = b$

$\Rightarrow \quad ax - ay - 1 = 0$

$$ax - a\left(1 - \frac{2x}{3}\right) - 1$$

$$x\left(a + \frac{2a}{3}\right) = \frac{a}{3}$$

$$x = \frac{a+3}{5a}$$

$$2\left(\frac{a+3}{5a}\right) + 3y - 1 = 0$$

$$y = \frac{1 - \frac{2a+6}{5a}}{3} = \frac{3a-6}{3\times 5a}$$

$$y = \frac{a-2}{5a}$$

$$\frac{\left(\frac{a-2}{5a}\right)}{\left(\frac{a+3}{5a}\right)} = 2 \Rightarrow a - 2 = 2a + 6$$

$a = -8$

$b = 8$

$-8x + 8y - 1 = 0$

$|a - b| = 16$

23. Three balls are drawn at random from a bag containing 5 blue and 4 yellow balls. Let the random variables X and Y respectively denote the number of blue and Yellow balls. If $\bar{X}$ and $\bar{Y}$ are the means of X and Y respectively, then $7\bar{X} + 4\bar{Y}$ is equal to _______.

Ans. (17)

Sol.

Blue balls	0	1	2	3	4	5
Prob.	$\frac{^5C_0 \cdot {}^4C_1}{^9C_3}$	$\frac{^5C_1 \cdot {}^4C_2}{^9C_3}$	$\frac{^5C_2 \cdot {}^4C_1}{^9C_3}$	$\frac{^5C_3 \cdot {}^4C_0}{^9C_3}$	0	0

$$7\bar{x} = \frac{^5C_1 {}^4C_2 + {}^5C_2 \cdot {}^4C_1 \times 2 + {}^5C_3 \cdot {}^4C_0 \times 3}{^9C_3} \times 7$$

$$\frac{30+80+30}{84} \times 7$$

$$= \frac{140}{12} = \frac{70}{6} = \frac{35}{3}$$

yellow	0	1	2	3	4
		$^5C_2 {}^4C_1$	$^5C_1 {}^4C_2$	$^5C_0 {}^4C_3$	0

$$4\bar{y} = \frac{40+60+12}{84} \times 4 = \frac{112}{21} = \frac{16}{3}$$

24. The number of 3-digit numbers, formed using the digits 2, 3, 4, 5 and 7, when the repetition of digits is not allowed, and which are not divisible by 3, is equal to _______.

Ans. (36)

Sol. **2, 3, 4, 5, 7**

total number of three digit numbers not divisible by 3 will be formed by using the digits

(4, 5, 7)
(3, 4, 7)
(2, 5, 7)
(2, 4, 7)
(2, 4, 5)
(2, 3, 5)

number of ways = 6 × 3! = 36

25. Let the positive integers be written in the form :

1

2 3

4 5 6

7 8 9 10

If the k^{th} row contains exactly k numbers for every natural number k, then the row in which the number 5310 will be, is ______.

Ans. (103)

Sol. $S = 1 + 2 + 4 + 7 + \ldots.. + T_n$

$S = 1 + 2 + 4 + \ldots\ldots$

$Tn = 1 + 1 + 2 + 3 + \ldots\ldots + (T_n - T_{n-1})$

$T_n = 1 + \left(\frac{n-1}{2}\right)[2 + (n-2)\times 1]$

$T_n = 1 + 1 + \frac{n(n-1)}{2}$

$n = 100 \qquad T_n = 1 + \frac{100\times 99}{2} = 4950 + 1$

$n = 101 \qquad T_n = 1 + \frac{101\times 100}{2} = 5050 + 1 = 5051$

$n = 102 \qquad T_n = 1 + \frac{102\times 101}{2} = 5151 + 1 = 5152$

$n = 103 \qquad T_n = 1 + \frac{103\times 102}{2} = 5254$

$n = 104 \qquad T_n = 1 + \frac{104\times 103}{2} = 5357$

26. If the range of $f(\theta) = \frac{\sin^4\theta + 3\cos^2\theta}{\sin^4\theta + \cos^2\theta}, \theta \in \mathbb{R}$ is $[\alpha, \beta]$, then the sum of the infinite G.P., whose first term is 64 and the common ratio is $\frac{\alpha}{\beta}$, is equal to ______.

Ans. (96)

Sol. $f(\theta) = \frac{\sin^4\theta + 3\cos^2\theta}{\sin^4\theta + \cos^2\theta}$

$f(\theta) = 1 + \frac{2\cos^2\theta}{\sin^4\theta + \cos^2\theta}$

$f(\theta) = \frac{2\cos^2\theta}{\cos^4\theta - \cos^2\theta + 1} + 1$

$f(\theta) = \frac{2}{\cos^2\theta + \sec^2\theta - 1} + 1$

$f(\theta)|_{min.} = 1$

$f(\theta)_{max.} = 3$

$S = \frac{64}{1 - 1/3} = 96$

27. Let $\alpha = \sum_{r=0}^{n}(4r^2 + 2r + 1)\,{}^nC_r$

and $\beta = \left(\sum_{r=0}^{n}\frac{{}^nC_r}{r+1}\right) + \frac{1}{n+1}$. If $140 < \frac{2\alpha}{\beta} < 281$,

then the value of n is ______.

Ans. (5)

Sol. $\alpha = \sum_{r=0}^{n}(4r^2 + 2r + 1).{}^nC_r$

$\alpha = 4\sum_{r=0}^{n} r^2.\frac{n}{r}.{}^{n-1}C_{r-1} + 2\sum_{r=0}^{n} r.\frac{n}{r}.{}^{n-1}C_{r-} + \sum_{r=0}^{n}{}^nC_r$

$+4n\sum_{r=0}^{n}{}^{n-1}C_{r-1} + 2n\sum_{r=0}^{n}{}^{n-1}C_{r-1} + \sum_{r=0}^{n}{}^nC_r$

$\alpha = 4n(n-1) . 2^{n-2} + 4n.2^{n-1} + 2n.2^{n-1} + 2^n$

$\alpha = 2^{n-2}[4n(n-1) + 8n + 4n + 4]$

$\alpha = 2^{n-2}[4n^2 + 8n + 4]$

$\alpha = 2n(n + 1)^2$

$\beta = \sum_{r=0}^{n}\frac{{}^nC_r}{r+1} + \frac{1}{n+1}$

$= \sum_{r=0}^{n}\frac{{}^{n+1}C_{r+1}}{n+1} + \frac{1}{n+1}$

$= \frac{1}{n+1}\left(1 + {}^{n+1}C_1 + \ldots. + {}^{n+1}C_{n+1}\right)$

$= \frac{2^{n+1}}{n+1}$

$\frac{2\alpha}{\beta} = \frac{2^{n+1}(n+1)^2}{2^{n+1}}.(n+1) = (n+1)^3$

$140 < (n + 1)^3 < 281$

$n = 4 \Rightarrow (n + 1)^3 = 125$

$n = 5 \Rightarrow (n + 1)^3 = 216$

$n = 6 \Rightarrow (n + 1)^3 = 343$

$\therefore n = 5$

28. Let $\vec{a}=9\hat{i}-13\hat{j}+25\hat{k}$, $\vec{b}=3\hat{i}+7\hat{j}-13\hat{k}$ and $\vec{c}=17\hat{i}-2\hat{j}+\hat{k}$ be three given vectros. If $\vec{r}$ is a vector such that $\vec{r}\times\vec{a}=(\vec{b}+\vec{c})\times\vec{a}$ and $\vec{r}.(\vec{b}-\vec{c})=0$, then $\dfrac{|593\vec{r}+67\vec{a}|^2}{(593)^2}$ is equal to ________ .

Ans. (569)

Sol. $\vec{a}=9\hat{i}-13\hat{j}+25\hat{k}$

$\vec{b}=3\hat{i}+7\hat{j}-13\hat{k}$

$\vec{c}=17\hat{i}-2\hat{j}+\hat{k}$

$\vec{b}+\vec{c}=20\hat{i}+5\hat{j}-12\hat{k}$

$\vec{b}-\vec{c}=-14\hat{i}+9\hat{j}-14\hat{k}$

$\left(\vec{r}-(\vec{b}+\vec{c})\right)\times\vec{a}=0$

$r-(\vec{b}+\vec{c})=\lambda\vec{a}$

$\vec{r}=\lambda\vec{a}+\vec{b}+\vec{c}$

But $\vec{r}.(\vec{b}-\vec{c})=0$

$\Rightarrow (\lambda\vec{a}+\vec{b}+\vec{c}).(\vec{b}-\vec{c})=0$

$\Rightarrow \lambda\vec{a}.\vec{b}+\vec{b}.\vec{b}+\vec{c}.\vec{b}-\lambda\vec{a}.\vec{c}-\vec{b}.\vec{c}-\vec{c}.\vec{c}=0$

$\lambda=\dfrac{\vec{c}.\vec{c}-\vec{b}.\vec{b}}{\vec{a}.\vec{b}-\vec{a}.\vec{c}}=\dfrac{294-227}{-389=204}=\dfrac{-67}{593}$

$\therefore\ \vec{r}=\vec{b}+\vec{c}-\dfrac{67}{593}\vec{a}$

$\Rightarrow 593\vec{r}+67\vec{a}=593(\vec{b}+\vec{c})$

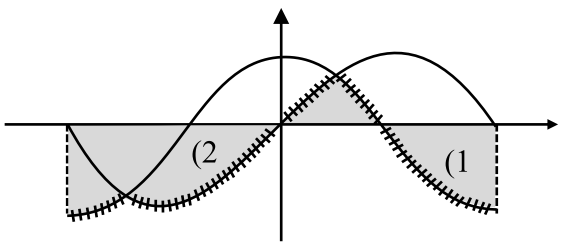
$\Rightarrow |\vec{b}+\vec{c}|^2=569$

29. Let the area of the region enclosed by the curve $y = \min\{\sin x, \cos x\}$ and the x-axis between $x = -\pi$ to $x = \pi$ be A. Then A^2 is equal to ________.

Ans. (16)

Sol. $y = \min\{\sin x, \cos x\}$

x–axis $x-\pi$ $x=\pi$



$\int_0^{\pi/4}\sin x=(\cos x)^0_{\pi/4}=1-\dfrac{1}{\sqrt{2}}$

$\int_{-\pi}^{-3\pi/4}(\sin x-\cos x)=(-\cos x-\sin x)^{-3\pi/4}_{-\pi}$

$=(\cos x+\sin x)^{-\pi}_{-3\pi/4}$

$=(-1+0)-\left(-\dfrac{1}{\sqrt{2}}-\dfrac{1}{\sqrt{2}}\right)$

$=-1+\dfrac{1}{\sqrt{2}}+\dfrac{1}{\sqrt{2}}$

$\int_{\pi/4}^{\pi/2}\cos x dx=(\sin x)^{\pi/2}_{\pi/4}=1-\dfrac{1}{\sqrt{2}}$

A = 4

$A^2 = 16$

30. The value of

$\lim\limits_{x\to0}2\left(\dfrac{1-\cos x\sqrt{\cos 2x}\sqrt[3]{\cos 3x}.....\sqrt[10]{\cos 10x}}{x^2}\right)$ is ________.

Ans. (55)

Sol.

$$\lim_{x\to0}2\left(\frac{1-\left(1-\frac{x^2}{2!}\right)\left(1-\frac{4x^2}{2!}\right)\left(1-\frac{9x^2}{2!}\right).....\left(1-\frac{100x^2}{2!}\right)}{x^2}\right)$$

By expansion

$$\lim_{x\to0}\frac{2\left(1-\left(1-\frac{x^2}{2}\right)\right)\left(1-\frac{1}{2}.\frac{4x^2}{2}\right)\left(1-\frac{1}{3}.\frac{9x^2}{2}\right).....\left(1-\frac{1}{10}.\frac{100x^2}{2}\right)}{x^2}$$

$$\lim_{x\to0}2\left(\frac{1-\left(1-\frac{x^2}{2}\right)\left(1-\frac{2x^2}{2}\right)\left(1-\frac{3x^2}{2}\right).....\left(1-\frac{10x^2}{2}\right)}{x^2}\right)$$

$$\lim_{x\to0}\frac{2\left(1-1+x^2\left(\frac{1}{2}+\frac{2}{2}+\frac{3}{2}+.....+\frac{10}{2}\right)\right)}{x^2}$$

$$2\left(\frac{1}{2}+\frac{2}{2}+\frac{3}{2}+.....+\frac{10}{2}\right)$$

$$1+2+......+10=\frac{10\times11}{2}=55$$