FINAL JEE–MAIN EXAMINATION – APRIL, 2024

(Held On Tuesday 09th April, 2024) **TIME : 3 : 00 PM to 6 : 00 PM**

MATHEMATICS | **TEST PAPER WITH SOLUTION**

SECTION-A

1. $\lim_{x\to 0} \frac{e-(1+2x)^{\frac{1}{2x}}}{x}$ is equal to :

(1) e (2) $\frac{-2}{e}$

(3) 0 (4) $e-e^2$

Ans. (1)

Sol. $\lim_{x\to 0} \frac{e-e^{\frac{1}{2x}\ln(1+2x)}}{x}$

$= \lim_{x\to 0}(-e)\frac{\left(e^{\frac{\ln(1+2x)}{2x}-1}-1\right)}{x}$

$= \lim_{x\to 0}(-e)\frac{\ln(1+2x)-2x}{2x^2}$

$= (-e)\times(-1)\ \frac{4}{2\times 2} = e$

2. Consider the line L passing through the points (1, 2, 3) and (2, 3, 5). The distance of the point $\left(\frac{11}{3},\frac{11}{3},\frac{19}{3}\right)$ from the line L along the line $\frac{3x-11}{2}=\frac{3y-11}{1}=\frac{3z-19}{2}$ is equal to :

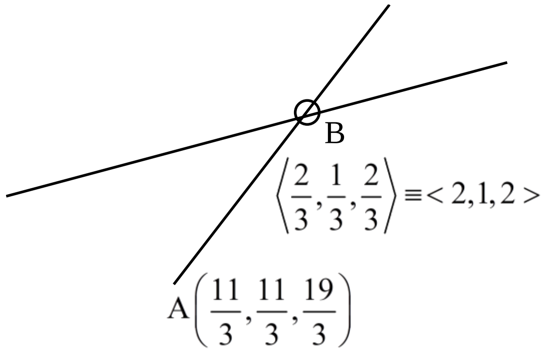
(1) 3 (2) 5

(3) 4 (4) 6

Ans. (1)

Sol. $\frac{x-1}{2-1}=\frac{y-2}{3-2}=\frac{z-3}{5-3}$

$\Rightarrow \frac{x-1}{1}=\frac{y-2}{1}=\frac{z-3}{2}=\lambda$



$B(1+\lambda, 2+\lambda, 3+2\lambda)$

D.R. of AB $= <\frac{3\lambda-8}{3},\frac{3\lambda-5}{3},\frac{6\lambda-10}{3}>$

$B\left(\frac{5}{3},\frac{8}{3},\frac{13}{3}\right)$ $\frac{3\lambda-8}{3\lambda-5}=\frac{2}{1} \Rightarrow 3\lambda-8=6\lambda-10$

$3\lambda = 2$

$\lambda = \frac{2}{3}$

$AB = \frac{\sqrt{36+9+36}}{3}=\frac{9}{3}=3$

3. Let $\int_0^x \sqrt{1-(y'(t))^2}\,dt = \int_0^x y(t)dt$, $0 \le x \le 3$, $y \ge 0$, y (0) = 0. Then at x = 2, y'' + y + 1 is equal to :

(1) 1 (2) 2

(3) $\sqrt{2}$ (4) 1/2

Ans. (1)

Sol. $\sqrt{1-(y'(x))^2}=y(x)$

$1-\left(\frac{dy}{dx}\right)^2=y^2$

$\left(\frac{dy}{dx}\right)^2=1-y^2$

$$\frac{dy}{\sqrt{1-y^2}} = dx \text{ OR } \frac{dy}{\sqrt{1-y^2}} = -dx$$

$\Rightarrow \sin^{-1}y = x + c, \sin^{-1}y = -x + c$

$x = 0, y = 0 \quad \Rightarrow \quad c = 0$

$\sin^{-1} y = x$, as $y \geq 0$

$\sin x = y$

$$\Rightarrow \frac{dy}{dx} = \cos x$$

$$\frac{d^2y}{dx^2} = -\sin x$$

$\Rightarrow - \sin x + \sin x + 1 = 1$

4. Let z be a complex number such that the real part of $\frac{z-2i}{z+2i}$ is zero. Then, the maximum value of $|z-(6+8i)|$ is equal to :

(1) 12 (2) ∞

(3) 10 (4) 8

Ans. (1)

Sol. $\frac{z-2i}{z+2i} + \frac{\bar{z}+2i}{\bar{z}-2i} = 0$

$z\bar{z} - 2i\bar{z} - 2iz + 4(-1)$

$+z\bar{z} + 2zi + 2\bar{z}i + 4(-1) = 0$

$\Rightarrow 2|z|^2 = 8 \Rightarrow |z| = 2$

$|z-(6+8i)|_{\text{maximum}} = 10 + 2 = 12$

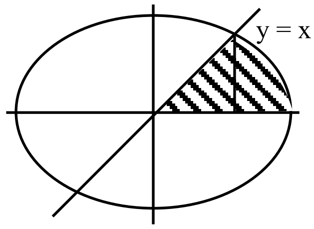
5. The area (in square units) of the region enclosed by the ellipse $x^2 + 3y^2 = 18$ in the first quadrant below the line $y = x$ is :

(1) $\sqrt{3}\pi + \frac{3}{4}$ (2) $\sqrt{3}\pi$

(3) $\sqrt{3}\pi - \frac{3}{4}$ (4) $\sqrt{3}\pi + 1$

Ans. (2)

Sol. $\frac{x^2}{18} + \frac{y^2}{6} = 1$



$$\frac{x^2}{18} + \frac{3x^2}{18} = 1 \Rightarrow 4x^2 = 18 \Rightarrow x^2 = \frac{9}{2}$$

$$\int_{\frac{3}{\sqrt{2}}}^{3\sqrt{2}} \frac{\sqrt{18-x^2}}{\sqrt{3}} dx$$

$$= \frac{1}{\sqrt{3}}\left(\frac{x\sqrt{18-x^2}}{2} + \frac{18}{2}\sin^{-1}\frac{x}{3\sqrt{2}}\right)_{\frac{3}{\sqrt{2}}}^{3\sqrt{2}}$$

$$= \frac{1}{\sqrt{3}}\left(9\times\frac{\pi}{2} - \frac{3}{2\sqrt{2}}\times\frac{3\sqrt{3}}{\sqrt{2}} - 9\times\frac{\pi}{6}\right)$$

Required Area

$$= \frac{1}{2}\times\frac{9}{2} + \left(\frac{18\pi}{6} - \frac{9\sqrt{3}}{4}\right)\frac{1}{\sqrt{3}}$$

$$= \sqrt{3}\pi$$

6. Let the foci of a hyperbola H coincide with the foci of the ellipse E : $\frac{(x-1)^2}{100} + \frac{(y-1)^2}{75} = 1$ and the eccentricity of the hyperbola H be the reciprocal of the eccentricity of the ellipse E. If the length of the transverse axis of H is α and the length of its conjugate axis is β, then $3\alpha^2 + 2\beta^2$ is equal to :

(1) 242

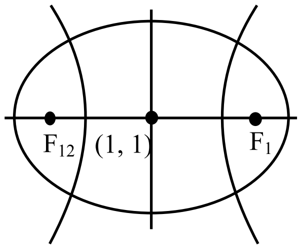
(2) 225

(3) 237

(4) 205

Ans. (2)

Sol.



$e_1 = \sqrt{1-\frac{75}{100}} = \frac{5}{10} = \frac{1}{2}$

$e_2 = 2$

F_1 (6, 1), F_2 (–4, 1)

$2ae_2 = 10 \Rightarrow a = \frac{5}{2} \Rightarrow 2a = 5$

$\Rightarrow \alpha = 5$

$4 = 1 + \frac{b^2}{a^2} \Rightarrow b^2 = 3a^2$

$b = \sqrt{3} \times \frac{5}{2}$

$\beta = 5\sqrt{3}$

$3\alpha^2 + 2\beta^2 = 3 \times 25 + 2 \times 25 \times 3$

$= 225$

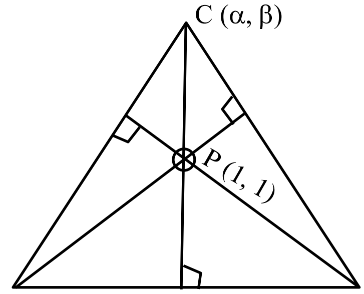
7. Two vertices of a triangle ABC are A(3, –1) and B (–2, 3), and its orthocentre is P(1, 1). If the coordinates of the point C are (α, β) and the centre of the circle circumscribing the triangle PAB is (h, k), then the value of (α + β) + 2 (h + k) equals :

(1) 51 (2) 81

(3) 5 (4) 15

Ans. (3)

Sol.



$M_{AB} = \frac{4}{-5} \Rightarrow M_{DP} = \frac{5}{4}$

Equation of PC is $y - 1 = \frac{5}{4}(x-1)$ ....(1)

$M_{AP} = \frac{2}{-2} = -1 \Rightarrow M_{BC} = +1$

Equation of BC is y – 3 = (x + 2) ....(2)

On solving (1) and (2)

$x + 4 = \frac{5}{4}(x-1) \Rightarrow 4x + 16 = 5x - 5 \Rightarrow \alpha = 21$

$\Rightarrow \beta = y = x + 5 = 26$

$\alpha + \beta = 47$

Equation of ⊥ bisector of AP

y – 0 = (x – 2) .... (3)

Equation of ⊥ bisector of AB

$y - 1 = \frac{5}{4}\left(x - \frac{1}{2}\right)$ ....(4)

On solving (3) & (4)

$(x - 3)4 = 5x - \frac{5}{2}$

$x = \frac{-19}{2} = h$

$y = \frac{-23}{2} = k$

$\Rightarrow 2(h + k) = -42$

8. If the variance of the frequency distribution is 160, then the value of $c \in N$ is

x	c	2c	3c	4c	5c	6c
f	2	1	1	1	1	1

(1) 5 (2) 8

(3) 7 (4) 6

Ans. (3)

Sol.

x	C	2C	3C	4C	5C	6C
f	2	1	1	1	1	1

$\bar{x} = \frac{(2+2+3+4+5+6)C}{7} = \frac{22C}{7}$

$\text{Var}(x) = \dfrac{c^2(2+2^2+3^2+4^2+5^2+6^2)}{7}$

$-\left(\dfrac{22c}{7}\right)^2$

$= \dfrac{92c^2}{7} - c^2 \times \dfrac{484}{49}$

$= \dfrac{(644-484)c^2}{49} = \dfrac{160c^2}{49}$

$160 = \dfrac{160 \times c^2}{49} \Rightarrow c = 7$

9. Let the range of the function

$f(x) = \dfrac{1}{2+\sin 3x + \cos 3x}$, $x \in \text{IR}$ be [a, b].

If α and β are respectively the A.M. and the G.M. of a and b, then $\dfrac{\alpha}{\beta}$ is equal to :

(1) $\sqrt{2}$ (2) 2

(3) $\sqrt{\pi}$ (4) π

Ans. (1)

Sol. $f(x) \dfrac{1}{2+\sin 3x+\cos 3x}$

$\left[\dfrac{1}{2+\sqrt{2}}, \dfrac{1}{2-\sqrt{2}}\right]$

$\dfrac{\alpha}{\beta} = \dfrac{a+b}{2\sqrt{ab}} = \dfrac{1}{2}\left(\sqrt{\dfrac{a}{b}} + \sqrt{\dfrac{b}{a}}\right)$

$= \dfrac{1}{2}\left(\sqrt{\dfrac{2-\sqrt{2}}{2+\sqrt{2}}} + \sqrt{\dfrac{2+\sqrt{2}}{2-\sqrt{2}}}\right)$

$= \dfrac{(2-\sqrt{2})+(2+\sqrt{2})}{2\times\sqrt{2}} = \sqrt{2}$

10. Between the following two statements :

Statement-I : Let $\vec{a} = \hat{i}+2\hat{j}-3\hat{k}$ and $\vec{b} = 2\hat{i}+\hat{j}-\hat{k}$. Then the vector $\vec{r}$ satisfying $\vec{a}\times\vec{r} = \vec{a}\times\vec{b}$ and $\vec{a}.\vec{r} = 0$ is of magnitude $\sqrt{10}$.

Statement-II : In a triangle ABC, cos2A + cos2B + cos2C $\geq -\dfrac{3}{2}$.

(1) Both Statement-I and Statement-II are incorrect

(2) Statement-I is incorrect but Statement-II is correct

(3) Both Statement-I and Statement-II are correct

(4) Statement-I is correct but Statement-II is incorrect

Ans. (2)

Sol. $\bar{a} = \hat{i}+2\hat{j}-3\hat{k}$

$\bar{a} = 2\hat{i}+\hat{j}-\hat{k}$

$\bar{a}\times\bar{r} = \bar{a}\times\bar{b}$; $\bar{a}.\bar{r} = 0$

$\Rightarrow \bar{a}\times(\bar{r}-\bar{b}) = \bar{0}$

$\Rightarrow \bar{a} = \lambda(\bar{r}-\bar{b})$

$\bar{a}.\bar{a} = \lambda(\bar{a}.\bar{r} - \bar{a}.\bar{b})$

$14 = -7\lambda \Rightarrow \lambda = -2$

$\dfrac{-\bar{a}}{2} = \bar{r}-\bar{b} \Rightarrow \bar{r} = \bar{b}-\dfrac{\bar{a}}{2}$

$= \dfrac{2\bar{b}-\bar{a}}{2} = \dfrac{3\hat{i}+\hat{k}}{2}$

Statement (I) is incorrect

cos2A + cos 2B + cos 2c $\geq -\dfrac{3}{2}$

$2A + 2B + 2C = 2\pi$

cos2A + cos2B + cos 2C

$= -1 - 4\cos A\cdot\cos B\cdot\cos C$

$\geq -1 - 4 \times \dfrac{1}{2}\times\dfrac{1}{2}\times\dfrac{1}{2}$

$= -\dfrac{3}{2}$

Statement (II) is correct.

11. $\lim\limits_{x \to \frac{x}{2}} \left(\dfrac{\int_{x^3}^{(\pi/2)^3} \left(\sin\left(2t^{1/3}\right) + \cos\left(t^{1/3}\right)\right) dt}{\left(x - \frac{\pi}{2}\right)^2} \right)$ is equal to :

(1) $\dfrac{9\pi^2}{8}$ (2) $\dfrac{11\pi^2}{10}$

(3) $\dfrac{3\pi^2}{2}$ (4) $\dfrac{5\pi^2}{9}$

Ans. (1)

Sol. $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{0 - \{\sin(2x) + \cos(x)\}.3x^2}{2\left(x - \frac{\pi}{2}\right)}$

$= \lim\limits_{x \to \frac{\pi}{2}} \dfrac{-\{2\sin x \cos x + \cos x\} 3x^2}{2\left(x - \frac{\pi}{2}\right)}$

$= \lim\limits_{x \to \frac{\pi}{2}} \left\{ \dfrac{2\sin x \sin\left(\frac{\pi}{2} - x\right)}{2\left(x - \frac{\pi}{2}\right)} + \dfrac{\sin\left(\frac{\pi}{2} - x\right)}{2\left(\frac{\pi}{2} - x\right)} \right\} 3x^2$

$= \left(1(1) + \frac{1}{2}\right) 3 \left(\frac{\pi}{2}\right)^2$

$= \dfrac{9\pi^2}{8}$

12. The sum of the coefficient of $x^{2/3}$ and $x^{-2/5}$ in the binomial expansion of $\left(x^{2/3} + \frac{1}{2}x^{-2/5}\right)^9$ is :

(1) 21/4 (2) 69/16

(3) 63/16 (4) 19/4

Ans. (1)

Sol. $T_{r+1} = {}^9C_r \left(x^{2/3}\right)^{9-r} \left(\dfrac{x^{-2/5}}{2}\right)^r$

$= {}^9C_r \left(\dfrac{1}{2}\right)^r (r)^{\left(6 - \frac{2r}{3} - \frac{2r}{5}\right)}$

for coefficient of $x^{2/3}$, put $6 - \dfrac{2r}{3} - \dfrac{2r}{5} = \dfrac{2}{3}$

$\Rightarrow r = 5$

$\therefore$ Coefficient of $x^{2/3}$ is $= {}^9C_5 \left(\dfrac{1}{5}\right)^5$

For coefficient of $x^{-2/5}$, put $6 - \dfrac{2r}{3} - \dfrac{2r}{5} = -\dfrac{2}{5}$

$\Rightarrow r = 6$

Coefficient of $x^{-2/5}$ is ${}^9C_6 \left(\dfrac{1}{2}\right)^6$

Sum $= {}^9C_5 \left(\dfrac{1}{2}\right)^5 + {}^9C_6 \left(\dfrac{1}{2}\right)^6 = \dfrac{21}{4}$

13. Let $B = \begin{bmatrix} 1 & 3 \\ 1 & 5 \end{bmatrix}$ and A be a 2 × 2 matrix such that $AB^{-1} = A^{-1}$. If $BCB^{-1} = A$ and $C^4 + \alpha C^2 + \beta I = O$, then $2\beta - \alpha$ is equal to :

(1) 16 (2) 2

(3) 8 (4) 10

Ans. (4)

Sol. $BCB^{-1} = A$

$\Rightarrow (BCB^{-1})(BCB^{-1}) = A.A$

$\Rightarrow BCI\,CB^{-1} = A^2$

$\Rightarrow BC^2B^{-1} = A^2$

$\Rightarrow B^{-1}(BC^2B^{-1})B = B^{-1}(A.A)B$

From equation (1)

$C^2 = A^{-1}.A.B$

$C^2 = B$

Also $AB^{-1} = A^{-1}$

$\Rightarrow AB^{-1}.A = A^{-1} A = I$

$\Rightarrow A^{-1}(AB^{-1}A) = A^{-1}I$

$B^{-1}A = A^{-1}$

Now characteristics equation of C^2 is

$|C_2 - \lambda I| = 0$

$|B - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 1-\lambda & 3 \\ 1 & 5-\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)(5-l) - 3 = 0 \Rightarrow (\lambda^2 - 6\lambda + 5) - 3 = 0$

$\Rightarrow \lambda^2 - 6\lambda + 2 = 0$

$\Rightarrow \beta^2 - 6B + 2I = 0$

$\Rightarrow C^4 - 6C^2 + 2I = 0$

$\alpha = -6$

$\beta = 2$

$\therefore 2\beta - \alpha = 4 + 6 = 10$

14. If $\log_e y = 3\sin^{-1}x$, then $(1-x)^2 y'' - xy'$ at $x = \frac{1}{2}$ is equal to :

(1) $9e^{\pi/6}$ (2) $3e^{\pi/6}$

(3) $3e^{\pi/2}$ (4) $9e^{\pi/2}$

Ans. (4)

Sol. $\ln(y) = 3\sin^{-1}x$

$\frac{1}{y}\cdot y' = 3\left(\frac{1}{\sqrt{1-x^2}}\right)$

$\Rightarrow y' = \frac{3y}{\sqrt{1-x^2}}$ at $x = \frac{1}{2}$

$\Rightarrow y' = \frac{3e^{3\left(\frac{\pi}{6}\right)}}{\frac{\sqrt{3}}{2}} = 2\sqrt{3}e^{\frac{\pi}{2}}$

$\Rightarrow y'' = 3\left(\frac{\sqrt{1-x^2}\,y' - y\frac{1}{2\sqrt{1-x^2}}(-2x)}{(1-x^2)}\right)$

$\Rightarrow (1-x^2)y'' = 3\left(3y + \frac{xy}{\sqrt{1-x^2}}\right)$

$\downarrow$ at $x = \frac{1}{2}$, $y = e^{3\sin^{-1}\left(\frac{1}{2}\right)} = e^{3\left(\frac{\pi}{6}\right)} = e^{\frac{\pi}{2}}$

$(1-x^2)y''\Big|_{\text{at } x=\frac{1}{2}} = 3\left(3e^{\frac{\pi}{2}} + \frac{\frac{1}{2}\left(e^{\frac{\pi}{2}}\right)}{\frac{\sqrt{3}}{2}}\right)$

$= 3e^{\frac{\pi}{2}}\left(3 + \frac{1}{\sqrt{3}}\right)$

$(1-x^2)y'' - xy'\Big|_{\text{at } x=\frac{1}{2}}$

$= 3e^{\frac{\pi}{2}}\left(3 + \frac{1}{\sqrt{3}}\right) - \frac{1}{2}\left(2\sqrt{3}e^{\frac{\pi}{2}}\right) = 9e^{\frac{\pi}{2}}$

15. The integral $\int_{1/4}^{3/4} \cos\left(2\cot^{-1}\sqrt{\frac{1-x}{1+x}}\right)dx$ is equal to:

(1) –1/2 (2) 1/4

(3) 1/2 (4) –1/4

Ans. (4)

Sol. $I = \int_{1/4}^{3/4} \cos\left(2\cot^{-1}\left(\sqrt{\frac{1-x}{1+x}}\right)dx\right)$

$\int_{1/4}^{3/4} \cos\left(2\left(\tan^{-1}\sqrt{\frac{1+x}{1+x}}\right)\right)dx$

$\int_{1/4}^{3/4} \frac{1-\tan^2\left(\tan^{-1}\sqrt{\frac{1+x}{1-x}}\right)}{1+\tan^2\left(\tan^{-1}\sqrt{\frac{1+x}{1-x}}\right)}dx$

$= \int_{1/4}^{3/4} \frac{1-\left(\frac{1+x}{1-x}\right)}{1+\left(\frac{1+x}{1-x}\right)}dx = \int_{1/4}^{3/4} \frac{-2x}{2}dx$

$= \int_{1/4}^{3/4} (-x)\,dx = -\left(\frac{x^2}{2}\right)_{1/4}^{3/4}$

$= -\frac{1}{2}\left[\frac{9}{16} - \frac{1}{16}\right]$

$= -\frac{1}{4}$

16. Let a, ar, ar^2, ...be an infinite G.P. If $\sum_{n=0}^{\infty} ar^n = 57$ and $\sum_{n=0}^{\infty} a^3r^{3n} = 9747$, then a + 18r is equal to :

(1) 27 (2) 46

(3) 38 (4) 31

Ans. (4)

Sol. $\sum_{n=0}^{\infty} ar^n = 57$

$a + ar + ar^2 + \infty = 57$

$\frac{a}{1-r} = 57$ .... (I)

$\sum_{n=0}^{\infty} a^3r^{3n} = 9747$

$a^3 + a^3 \cdot r^3 + a^3 \cdot r^6 + ...\infty = 9746$

$\frac{a^3}{1-r^3} = 9746$ .... (II)

$$\frac{(I)^3}{(II)} \Rightarrow \frac{\frac{a^3}{(1-r)^3}}{\frac{a^3}{1-r^3}} = \frac{57^3}{9717} = 19$$

On solving, $r = \frac{2}{3}$ and $r = \frac{3}{2}$ (rejected)

a = 19

$\therefore a + 18r = 19 + 18 \times \frac{2}{3} = 31$

17. If an unbiased dice is rolled thrice, then the probability of getting a greater number in the i^{th} roll than the number obtained in the $(i-1)^{th}$ roll, i = 2, 3, is equal to :

(1) 3/54 (2) 2/54

(3) 5/54 (4) 1/54

Ans. (3)

Sol. Favourable cases = ${}^6C_3$

Total out comes = 6^3

Probability of getting greater number than previous one = $\frac{{}^6C_3}{r^3} = \frac{20}{216} = \frac{5}{54}$

18. The value of the integral $\int_{-1}^{2} \log_e\left(x + \sqrt{x^2+1}\right)dx$ is :

(1) $\sqrt{5} - \sqrt{2} + \log_e\left(\frac{9+4\sqrt{5}}{1+\sqrt{2}}\right)$

(2) $\sqrt{2} - \sqrt{5} + \log_e\left(\frac{9+4\sqrt{5}}{1+\sqrt{2}}\right)$

(3) $\sqrt{5} - \sqrt{2} + \log_e\left(\frac{7+4\sqrt{5}}{1+\sqrt{2}}\right)$

(4) $\sqrt{2} - \sqrt{5} + \log_e\left(\frac{7+4\sqrt{5}}{1+\sqrt{2}}\right)$

Ans. (2)

Sol. $I = \int_{-1}^{2} 1.\log_e\left(x + \sqrt{x^2+1}\right)dx$

$$= x\log_e\left(x + \sqrt{x^2+1}\right) - \int_{-1}^{2}\left(\frac{1 + \frac{x}{\sqrt{x^2+1}}}{x + \sqrt{x^2+1}}\right)dx$$

$$= x\log_e\left(x + \sqrt{x^2+1}\right) - \int_{-1}^{2}\frac{x}{\sqrt{x^2+1}}dx$$

$$= x\log_e\left(x + \sqrt{x^2+1}\right) - \sqrt{x^2+1}\Big|_{-1}^{2}$$

$$= \left(2\log_e\left(2+\sqrt{5}\right) - \sqrt{5}\right)$$

$$-\left(-\log_e\left(-1+\sqrt{2}\right) - \sqrt{2}\right)$$

$$= \log_e\left(2+\sqrt{5}\right)^2 - \sqrt{5} + \log_e\left(\sqrt{2}-1\right) + \sqrt{2}$$

$$= \log_e\left(2+\sqrt{5}\right)^2 - \sqrt{5} + \log_e\left(\sqrt{2}-1\right) + \sqrt{2}$$

$$=\sqrt{2}-\sqrt{5}+\log_e\left(\frac{\left(2+\sqrt{5}\right)^2}{\sqrt{2}+1}\right)$$

$$=\sqrt{2}-\sqrt{5}+\log_e\left(\frac{9+4\sqrt{5}}{\sqrt{2}+1}\right)$$

19. Let α, β; α > β, be the roots of the equation $x^2-\sqrt{2}x-\sqrt{3}=0$. Let $P_n = \alpha^n-\beta^n$, $n \in N$. Then $\left(11\sqrt{3}-10\sqrt{2}\right) P_{10} + \left(11\sqrt{2}+10\right) P_{11} - 11P_{12}$ is equal to :

(1) $10\sqrt{2}P_9$

(2) $10\sqrt{3}P_9$

(3) $11\sqrt{2}P_9$

(4) $11\sqrt{3}P_9$

Ans. (2)

Sol. $x^2-\sqrt{2x}-\sqrt{3}=0 \left\langle\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right.$

$\alpha^{n+2}-\sqrt{2}\alpha^{n+1}-\sqrt{3}\alpha^n=0$

and $\beta^{n+2}-\sqrt{2}\beta^{n+1}-\sqrt{3}\beta^n=0$

Subtracting

$\left(\alpha^{n+2}-\beta^{n+2}\right)-\sqrt{2}\left(\alpha^{n+1}-\beta^{n+1}\right)-\sqrt{3}\left(\alpha^n-\beta^n\right)=0$

$\Rightarrow P_{n+2}-\sqrt{2}P_{n+1}-\sqrt{3}P_n=0$

Put $n=10$

$P_{12}-\sqrt{2}P_{11}-\sqrt{3}P_{10}=0$

$n=9$

$P_{11}-\sqrt{2}P_{10}-\sqrt{3}P_9=0$

$11\left(\sqrt{3}.P_{10}+\sqrt{2}P_{11}-P_{11}\right)-10\left(\sqrt{2}P_{10}-P_{11}\right)$

$=0-10\left(-\sqrt{3}P_9\right)=10\sqrt{3}P_9$

20. Let $\vec{a}=2\hat{i}+\alpha\hat{j}+\hat{k}$, $\vec{b}=-\hat{i}+\hat{k}$, $\vec{c}=\beta\hat{j}-\hat{k}$, where α and β are integers and αβ = –6. Let the values of the ordered pair (α, β) for which the area of the parallelogram of diagonals $\vec{a}+\vec{b}$ and $\vec{b}+\vec{c}$ is $\frac{\sqrt{21}}{2}$, be (α_1, β_1) and (α_2, β_2).

Then $\alpha_1^2+\beta_1^2-\alpha_2\beta_2$ is equal to

(1) 17 (2) 24

(3) 21 (4) 19

Ans. (4)

Sol. Area of parallelogram = $\frac{1}{2}|\vec{d}_1\times\vec{d}_2$

$A=\frac{1}{2}\left|(\vec{a}+\vec{b})\times(\vec{b}+\vec{c})\right|=\frac{\sqrt{21}}{2}$

so, $\vec{a}+\vec{b}=\hat{i}+\alpha\hat{j}+2\hat{k}$

$\vec{b}+\vec{c}=-\hat{i}+\beta\hat{j}$

$$\left(\vec{a}+\vec{b}\right)\times\left(\vec{b}+\vec{c}\right)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & \alpha & 2\\ -1 & \beta & 0\end{vmatrix}$$

$=\hat{i}(-2\beta)-\hat{j}(2)+\hat{k}(\beta+\alpha)$

$\left|(\vec{a}+\vec{b})\times(\vec{b}+\vec{c})\right|=\sqrt{4\beta^2+4+(\alpha+\beta)^2}=\sqrt{21}$

$4\beta^2+4+\alpha^2+\beta^2+2\alpha\beta=21$

$\alpha^2+5\beta^2-12=17$

$\alpha^2+5\beta^2=29$

and $\alpha\beta=-6$

and given $\alpha_i\beta$ are integers

so,

$\alpha=-3,\ \beta=2$

or

$\alpha=3,\ \beta=-2$

$(\alpha_1,\beta_1)=(-3,2)$

$(\alpha_2,\beta_2)=(3,-2)$

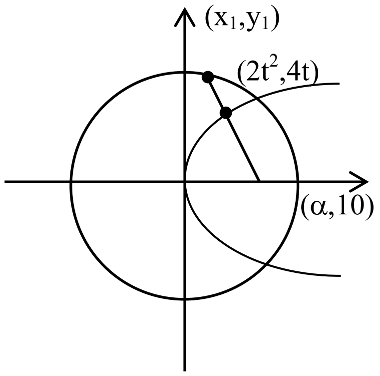
$\alpha_1^2+\beta_1^2-\alpha_2\beta_2=9+4+6=19$

SECTION-B

21. Consider the circle $C : x^2 + y^2 = 4$ and the parabola $P : y^2 = 8x$. If the set of all values of α, for which three chords of the circle C on three distinct lines passing through the point $(\alpha, 0)$ are bisected by the parabola P is the interval (p, q), then $(2q - p)^2$ is equal to _____.

Ans. (80)

Sol.



$T = S_1$

$xx_1 + yy_1 = x_1^2 + y_1^2$

$\alpha x_1 = x_1^2 + y_1^2$

$\alpha(2t^2) = 4t^4 + 16t^2$

$\alpha = 2t^2 + 8$

$\frac{\alpha - 8}{2} = t^2$

Also, $4t^4 + 16t^2 - 4 < 0$

$t^2 = -2 + \sqrt{5}$

$\alpha = 4 + 2\sqrt{5}$

$\therefore\ \alpha \in (8, 4 + 2\sqrt{5})$

$\therefore\ (2q - p)^2 = 80$

22. Let the set of all values of p, for which $f(x) = (p^2 - 6p + 8)(\sin^2 2x - \cos^2 2x) + 2(2 - p)x + 7$ does not have any critical point, be the interval (a, b). Then 16ab is equal to _____ .

Ans. (252)

Sol. $f(x) = -(p^2 - 6p + 8)\cos 4n + 2(2-p)n + 7$

$f^1(x) = +4(p^2 - 6p + 8)\sin 4x + (4-2p) \neq 0$

$\sin 4x \neq \frac{2p-4}{4(p-4)(p-2)}$

$\sin 4x \neq \frac{2(p-2)}{4(p-4)(p-2)}$

$p \neq 2$

$\sin 4x \neq \frac{1}{2(p-4)}$

$\Rightarrow \left|\frac{1}{2(p-4)}\right| > 1$

on solving we get

$\therefore\ p \in \left(\frac{7}{2}, \frac{9}{2}\right)$

Hence $a = \frac{7}{2}, b = \frac{9}{2}$

$\therefore\ 16ab = 252$

23. For a differentiable function $f : IR \to IR$, suppose $f'(x) = 3f(x) + \alpha$, where $\alpha \in$ IR, $f(0) = 1$ and $\lim_{x \to -\infty} f(x) = 7$. Then $9f(-\log_e 3)$ is equal to______.

Ans. (61)

Sol. $\frac{dy}{dx} - 3y = \alpha$

$\text{If} = e^{\int -3dx} = e^{-3x}$

$\therefore\ y - e^{-3x} = \int e^{-3x} \cdot \alpha\, dx$

$y\, e^{-3x} = \frac{\alpha\, e^{-3x}}{-3} + c$

$(*\ e^{3x})$

$y = \frac{\alpha}{-3} + C \cdot e^{3x}$

on substituting $x = 0, y = 1$

$x \to -\infty, y = 7$

we get $y = 7 - 6e^{3x}$

$9f(-\log_e 3) = 61$

24. The number of integers, between 100 and 1000 having the sum of their digits equals to 14, is ______.

Ans. (70)

Sol. N = a b c

(i) All distinct digits

$a + b + c = 14$

$a \geq 1$

$b, c \in \{0 \text{ to } 9\}$

by hit & trial : 8 cases

(6, 5, 3)	(8, 6, 0)	(9, 4, 1)
(7, 6, 1)	(8, 5, 1)	(9, 3, 2)
(7, 5, 2)	(8, 4, 2)	
(7, 4, 3)	(9, 5, 0)	

(ii) 2 same, 1 diff $a = b\ ;\ c$

$2a + c = 14$

by values :

$\left.\begin{matrix}(3,8)\\(4,6)\\(5,4)\\(6,2)\\(7,0)\end{matrix}\right\}$ Total $\frac{3!}{2!}\times 5 - 1$

= 14 cases

(iii) all same :

$3a = 14$

$a = \frac{14}{3}$ × rejected

0 cases

Hence, Total cases :

$8 \times 3! + 2 \times (4) + 14$

$= 48 + 22$

$= 70$

25. Let $A = \{(x, y) : 2x + 3y = 23,\ x, y \in \mathbb{N}\}$ and $B = \{x : (x, y) \in A\}$. Then the number of one-one functions from A to B is equal to _____.

Ans. (24)

Sol. $2x + 3y = 23$

$x = 1 \quad y = 7$

$x = 4 \quad y = 5$

$x = 7 \quad y = 3$

$x = 10 \quad y = 1$

A	B
(1, 7)	1
(4, 5)	4
(7, 3)	7
(10, 1)	10

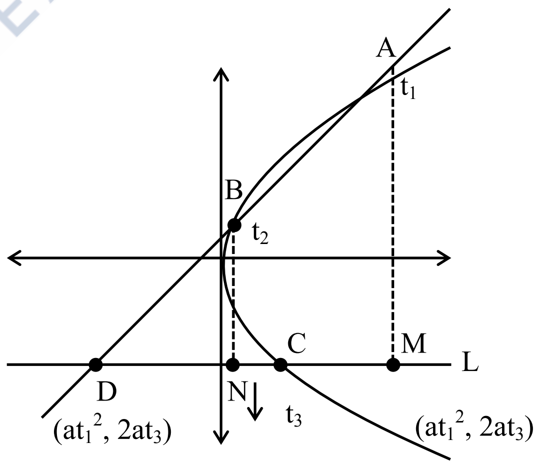
The number of one-one functions from A to B is equal to 4!

26. Let A, B and C be three points on the parabola $y^2 = 6x$ and let the line segment AB meet the line L through C parallel to the x-axis at the point D. Let M and N respectively be the feet of the perpendiculars from A and B on L.

Then $\left(\frac{AM \cdot BN}{CD}\right)^2$ is equal to _______.

Ans. (36)

Sol.



Sol.

$m_{AB} = m_{AD}$

$\Rightarrow \quad \frac{2}{t_1 + t_2} = \frac{2a(t_1 - t_3)}{at_1^2 - \alpha}$

$\Rightarrow \quad at_1^2 - \alpha = a\{t_1^2 - t_1t_3 + t_1t_2 - t_2t_3\}$

$\Rightarrow \quad \alpha = a(t_1t_3 + t_2t_3 - t_1t_2)$

$AM = |2a(t_1 - t_3)|$, $BN = |2a(t_2 - t_3)|$,

$CD = |at_3^2 - \alpha|$

$$CD = \left|at_3^2 - a(t_1t_3 + t_2t_3 - t_1t_2)\right|$$

$$= a\left|t_3^2 - t_1t_3 - t_2t_3 + t_1t_2\right|$$

$$= a\left|t_3(t_3 - t_1) - t_2(t_3 - t_1)\right|$$

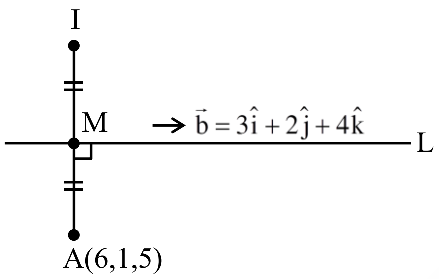
$$CD = a\left|(t_3 - t_2)(t_3 - t_1)\right|$$

$$\left(\frac{AM \cdot BN}{CD}\right)^2 = \left\{\frac{2a(t_1 - t_3)\cdot 2a(t_2 - t_3)}{a(t_3 - t_2)(t_3 - t_1)}\right\}^2$$

$$16a^2 = 16 \times \frac{9}{4} = 36$$

27. The square of the distance of the image of the point (6, 1, 5) in the line $\frac{x-1}{3} = \frac{y}{2} = \frac{z-2}{4}$, from the origin is ______.

Ans. (62)



Sol.

Let M(3λ + 1, 2λ, 4λ + 2)

$\overrightarrow{AM} \cdot \vec{b} = 0$

$\Rightarrow$ 9λ – 15 + 4λ – 2 + 16λ – 12 = 0

$\Rightarrow$ 29λ = 29

$\Rightarrow$ λ = 1

M (4, 2, 6), I = (2, 3, 7)

Required Distance = $\sqrt{4+9+49} = \sqrt{62}$

Ans. 62

28. If $\left(\frac{1}{\alpha+1} + \frac{1}{\alpha+2} + ... + \frac{1}{\alpha+1012}\right)$

$-\left(\frac{1}{2\cdot 1} + \frac{1}{4\cdot 3} + \frac{1}{6\cdot 5} + ... + \frac{1}{2024\cdot 2023}\right)$

$= \frac{1}{2024}$, then α is equal to-

Ans. (1011)

Sol. $\left(\frac{1}{\alpha+1} + \frac{1}{\alpha+2} + ... + \frac{1}{\alpha+2012}\right)$

$$-\left\{\left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + ... + \left(\frac{1}{2023} - \frac{1}{2024}\right)\right\} = \frac{1}{2024}$$

$$\Rightarrow \left(\frac{1}{\alpha+1} + \frac{1}{\alpha+2} + ... + \frac{1}{\alpha+2012}\right)$$

$$-\left\{\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) + ... + \frac{1}{2023}\right.$$

$$\left.-\frac{1}{2024} - 2\left(\frac{1}{2} + \frac{1}{4} + ... + \frac{1}{2022}\right)\right\} = \frac{1}{2024}$$

$$\Rightarrow \left(\frac{1}{\alpha+1} + \frac{1}{\alpha+2} + ... + \frac{1}{\alpha+2012}\right)$$

$$-\left(\frac{1}{1} + \frac{1}{2} + ... + \frac{1}{2023}\right)$$

$$+\frac{1}{2024} + \left(\frac{1}{1} + \frac{1}{2} + ... + \frac{1}{1011}\right) = \frac{1}{2024}$$

$$\Rightarrow \frac{1}{\alpha+1} + \frac{1}{\alpha+2} + ... + \frac{1}{\alpha+2012}$$

$$= \frac{1}{1012} + \frac{1}{1013} + ... + \frac{1}{2023}$$

$\Rightarrow$ α = 1011

29. Let the inverse trigonometric functions take principal values. The number of real solutions of the equation 2 $\sin^{-1}$ x + 3 $\cos^{-1}$ x = $\frac{2\pi}{5}$, is ______.

Ans. (0)

Sol. $2\sin^{-1}x + 3\cos^{-1}x = \frac{2\pi}{5}$

$\Rightarrow \quad \pi + \cos^{-1}x = \frac{2\pi}{5}$

$\Rightarrow \quad \cos^{-1}x = \frac{-3\pi}{5}$

Not possible

Ans. 0

30. Consider the matrices : $A = \begin{bmatrix} 2 & -5 \\ 3 & m \end{bmatrix}$, $B = \begin{bmatrix} 20 \\ m \end{bmatrix}$ and $X = \begin{bmatrix} x \\ y \end{bmatrix}$. Let the set of all m, for which the system of equations AX = B has a negative solution (i.e., $x < 0$ and $y < 0$), be the interval (a, b).

Then $8\int_a^b |A|\,dm$ is equal to ________.

Ans. (450)

Sol. $A = \begin{pmatrix} 2 & -5 \\ 3 & m \end{pmatrix}$, $B = \begin{pmatrix} 20 \\ m \end{pmatrix}$

$X = \begin{pmatrix} x \\ y \end{pmatrix}$

$2x - 5y = 20 \quad \ldots(1)$

$3x + my = m \quad \ldots(2)$

$\Rightarrow y = \dfrac{2m-60}{2m+15}$

$y < 0 \Rightarrow m \in \left(\dfrac{-15}{2}, 30\right)$

$x = \dfrac{25m}{2m+15}$

$x < 0 \Rightarrow m \in \left(\dfrac{-15}{2}, 0\right)$

$\Rightarrow m \in \left(\dfrac{-15}{2}, 0\right)$

$|A| = 2m + 15$

Now,

$$8\int_{-15/2}^{0}(2m+15)\,dm = 8\left\{m^2 + 15m\right\}_{-15/2}^{0}$$

$$\Rightarrow 8\left\{-\left(\frac{225}{4} - \frac{225}{2}\right)\right\}$$

$$= 8 \times \frac{225}{4} = 450$$