FINAL JEE–MAIN EXAMINATION – APRIL, 2024

(Held On Tuesday 09th April, 2024) **TIME : 9 : 00 AM to 12 : 00 NOON**

MATHEMATICS | TEST PAPER WITH SOLUTION

SECTION-A

1. Let the line L intersect the lines

$x - 2 = -y = z - 1,\ 2(x + 1) = 2(y - 1) = z + 1$

and be parallel to the line $\frac{x-2}{3}=\frac{y-1}{1}=\frac{z-2}{2}$.

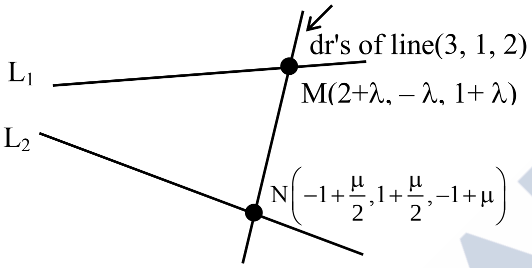
Then which of the following points lies on L ?

(1) $\left(-\frac{1}{3},1,1\right)$ (2) $\left(-\frac{1}{3},1,-1\right)$

(3) $\left(-\frac{1}{3},-1,-1\right)$ (4) $\left(-\frac{1}{3},-1,1\right)$

Ans. (2)

Sol.



$L_1 : \frac{x-2}{1}=\frac{y}{-1}=\frac{z-1}{1}=\lambda$

$L_2 : \frac{x+1}{\frac{1}{2}}=\frac{y-1}{\frac{1}{2}}=\frac{z+1}{1}=\mu$

dr of line MN will be

$<3+\lambda-\frac{\mu}{2},-1-\lambda-\frac{\mu}{2},2+\lambda-\mu>$ & it will be proportional to <3, 1, 2>

$\therefore \frac{3+\lambda-\frac{\mu}{2}}{3}=\frac{-1-\lambda-\frac{\mu}{2}}{1}=\frac{2+\lambda-\mu}{2}$

$4\lambda + \mu = -6$ $4 + 3\lambda = 0$

$\Rightarrow \lambda=-\frac{4}{3} \;\&\; \mu=-\frac{2}{3}$

$\therefore$ Coordinate of M will be $<\left(\frac{2}{3},\frac{4}{3},-\frac{1}{3}\right)$

and equation of required line will be.

$$\frac{x-\frac{2}{3}}{3}=\frac{y-\frac{4}{3}}{1}=\frac{z+\frac{1}{3}}{2}=k$$

So any point on this line will be

$$\left(\frac{2}{3}+3k,\frac{4}{3}+k,-\frac{1}{3}+2k\right)$$

$\therefore \frac{2}{3}+3k=-\frac{1}{3} \Rightarrow k=-\frac{1}{3}$

$\therefore$ Point lie on the line for

$k=-\frac{1}{3}$ is $\left(-\frac{1}{3},1,-1\right)$

2. The parabola $y^2 = 4x$ divides the area of the circle $x^2 + y^2 = 5$ in two parts. The area of the smaller part is equal to :

(1) $\frac{2}{3}+5\sin^{-1}\left(\frac{2}{\sqrt{5}}\right)$ (2) $\frac{1}{3}+5\sin^{-1}\left(\frac{2}{\sqrt{5}}\right)$

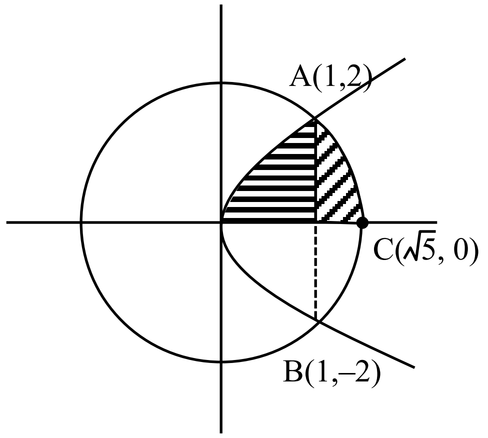
(3) $\frac{1}{3}+\sqrt{5}\sin^{-1}\left(\frac{2}{\sqrt{5}}\right)$ (4) $\frac{2}{3}+\sqrt{5}\sin^{-1}\left(\frac{2}{\sqrt{5}}\right)$

Ans. (1)

Sol. $y^2 = 4x$

$x^2 + y^2 = 5$

$\therefore$ Area of shaded region as shown in the figure will be



$$A_1 = \int_0^1 \sqrt{4x}\,dx + \int_1^{\sqrt{5}} \sqrt{5-x^2}\,dx$$

$$= \frac{4}{3}\cdot\left[x^{\frac{3}{2}}\right]_0^1 + \left[\frac{x}{2}\sqrt{5-x^2} + \frac{5}{2}\sin^{-1}\frac{x}{\sqrt{5}}\right]_1^{\sqrt{5}}$$

$$= \frac{1}{3} + \frac{5\pi}{4} - \frac{5}{2}\sin^{-1}\left(\frac{1}{\sqrt{5}}\right)$$

$\therefore$ Required Area = 2 A_1

$$= \frac{2}{3} + \frac{5\pi}{2} - 5\sin^{-1}\left(\frac{1}{\sqrt{5}}\right)$$

$$= \frac{2}{3} + 5\left(\frac{\pi}{2} - \sin^{-1}\frac{1}{\sqrt{5}}\right)$$

$$= \frac{2}{3} + 5\cos^{-1}\frac{1}{\sqrt{5}}$$

$$= \frac{2}{3} + 5\sin^{-1}\left(\frac{2}{\sqrt{5}}\right)$$

3. The solution curve, of the differential equation $2y\frac{dy}{dx} + 3 = 5\frac{dy}{dx}$, passing through the point (0, 1) is a conic, whose vertex lies on the line :

(1) $2x + 3y = 9$ (2) $2x + 3y = -9$

(3) $2x + 3y = -6$ (4) $2x + 3y = 6$

Ans. (1)

Sol. $(2y-5)\frac{dy}{dx} = -3$

$(2y-5)dy = -3dx$

$$2\cdot\frac{y^2}{2} - 5y = -3x + \lambda$$

$\because$ Curve passes through (0, 1)

$\Rightarrow \lambda = -4$

$\because$ Curve will be

$$\left(y - \frac{5}{2}\right)^2 = -3\left(x - \frac{3}{4}\right)$$

$\therefore$ Vertex of parabola will be $\left(\frac{3}{4}, \frac{5}{2}\right)$

$\because$ $2x + 3y = 9$

4. A ray of light coming from the point P (1, 2) gets reflected from the point Q on the x-axis and then passes through the point R (4, 3). If the point S (h, k) is such that PQRS is a parallelogram, then hk^2 is equal to :

(1) 80 (2) 90

(3) 60 (4) 70

Ans. (4)

Sol.

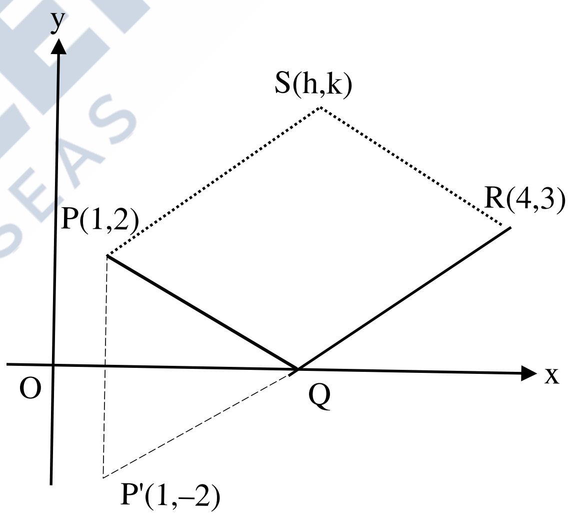


Image of P wrt x-axis will be P'(1, –2) equation of line joining P'R will be

$$y - 3 = \frac{5}{3}(x-4)$$

Above line will meet x-axis at Q where

$$y = 0 \Rightarrow x = \frac{11}{5}$$

$$\therefore Q\left(\frac{11}{5}, 0\right)$$

$\because$ PQRS is parallelogram so their diagonals will bisects each other

$$\Rightarrow \frac{4+1}{2}=\frac{\frac{11}{5}+h}{2} \;\&\; \frac{2+3}{2}=\frac{k+0}{2}$$

$$\Rightarrow h=\frac{14}{5} \;\&\; k=5$$

$$\therefore hk^2=\frac{14}{5}\times 5^2=70$$

5. Let $\lambda, \mu \in R$. If the system of equations

$3x + 5y + \lambda z = 3$

$7x + 11y - 9z = 2$

$97x + 155y - 189z = \mu$

has infinitely many solutions, then $\mu + 2\lambda$ is equal to :

(1) 25 (2) 24

(3) 27 (4) 22

Ans. (1)

Sol. $3x + 5y + \lambda z = 3$

$7x + 11y - 9z = 2$

$97x + 155y - 189z = \mu$

$93x + 155y + 31\lambda z = 93$

$97x + 155y - 189z = \mu$

$-\quad -\quad +\quad -$

$-4x + (31\lambda + 189)z = 93 - \mu$

$1085x + 1705y - 1395z = 310$

$1067x + 1705y - 2079z = 11\mu$

$-\quad -\quad +\quad -$

$18x + 684z = 310 - 11\mu$

$-36x + 9(31\lambda + 189)z = 9(93 - \mu)$

$36x + 1368z = 2(310 - 11\mu)$

$(279\lambda + 3069)z = 1457 - 31\mu$

for infinite solutions -

$$\lambda=\frac{-3069}{279}=\frac{-341}{31}$$

$$\mu=\frac{1457}{31}$$

$$\mu+2\lambda=\frac{1457-682}{31}=\frac{775}{31}=25$$

6. The coefficient of x^{70} in $x^2(1+x)^{98} + x^3(1+x)^{97} + x^4(1+x)^{96} + \ldots\ldots + x^{54}(1+x)^{46}$ is ${}^{99}C_p - {}^{46}C_q$.

Then a possible value to $p + q$ is :

(1) 55 (2) 61

(3) 68 (4) 83

Ans. (4)

Sol. $x^2(1+x)^{98}+x^3(1+x^{97})+x^4(1+x)^{96}+\ldots\ldots$

$x^{54}(1+x)^{46}$

Coeff. of x^{70} : ${}^{98}C_{68}+{}^{97}C_{67}+{}^{96}C_{66}+\ldots\ldots\ldots$

${}^{47}C_{17}+{}^{46}C_{16}$

$={}^{46}C_{30}+{}^{47}C_{30}+\ldots\ldots\ldots{}^{98}C_{30}$

$=\left({}^{46}C_{31}+{}^{46}C_{30}\right)+{}^{47}C_{30}+\ldots\ldots\ldots{}^{98}C_{30}-{}^{46}C_{31}$

$={}^{47}C_{31}+{}^{47}C_{30}+\ldots\ldots\ldots{}^{98}C_{30}-{}^{46}C_{31}$

......

$={}^{99}C_{31}-{}^{46}C_{31}={}^{99}C_p-{}^{46}C_q$

Possible values of $(p + q)$ are 62, 83, 99, 46

$\Rightarrow p + q = 83$

7. Let

$$\int\frac{2-\tan x}{3+\tan x}dx=\frac{1}{2}\left(\alpha x+\log_e\left|\beta\sin x+\gamma\cos x\right|\right)+C$$

, where C is the constant of integration.

Then $\alpha+\frac{\gamma}{\beta}$ is equal to :

(1) 3 (2) 1

(3) 4 (4) 7

Ans. (3)

Sol. $$\int\frac{2-\tan x}{3+\tan x}dx=\int\frac{2\cos x-\sin x}{3\cos x+\sin x}dx$$

$2\cos x - \sin x = A(3\cos x + \sin x) + B(\cos x - 3\sin x)$

$3A + B = 2$

$A - 3B = -1$

$\Rightarrow A = \frac{1}{2}, B = \frac{1}{2}$

$\therefore \int \frac{2\cos x - \sin x}{3\cos x + \sin x} dx$

$= \frac{x}{2} + \frac{1}{2}\ln|3\cos x + \sin x| + C$

$= \frac{1}{2}\left(x + \ln|3\cos x + \sin x|\right) + C$

$= \frac{1}{2}\left(\alpha x + \ln|\beta \sin x + \gamma \cos x|\right) + C$

$\alpha = 1, \beta = 1, \gamma = 3$

$\therefore \alpha + \frac{\gamma}{\beta} = 1 + \frac{3}{1} = 4$

8. A variable line L passes through the point (3, 5) and intersects the positive coordinate axes at the points A and B. The minimum area of the triangle OAB, where O is the origin, is :

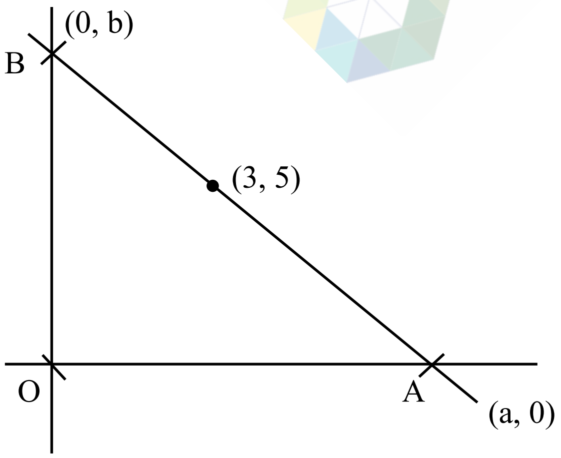
(1) 30 (2) 25

(3) 40 (4) 35

Ans. (1)

Sol. $\frac{x}{a} + \frac{y}{b} = 1$

$\frac{3}{a} + \frac{5}{b} = 1 \Rightarrow b = \frac{5a}{a-3}, a > 3$



$A = \frac{1}{2}ab = \frac{1}{2}a\frac{5a}{(a-3)} = \frac{5}{2}\cdot\frac{a^2}{a-3}$

$= \frac{5}{2}\left(\frac{a^2 - 9 + 9}{a-3}\right)$

$= \frac{5}{2}\left(a + 3 + \frac{9}{a-3}\right)$

$= \frac{5}{2}\left(a - 3 + \frac{9}{a-3} + 6\right) \geq 30$

9. Let

$|\cos\theta\cos(60-\theta)\cos(60-\theta)| \leq \frac{1}{8}, \theta \in [0, 2\pi]$

Then, the sum of all $\theta \in [0, 2\pi]$, where cos 3θ attains its maximum value, is :

(1) 9π (2) 18 π

(3) 6 π (4) 15 π

Ans. (3)

Sol. We know that

(cos θ) (cos (60° – θ) (cos (60° + θ) $= \frac{1}{4}\cos 3\theta$

So equation reduces to $\left|\frac{1}{4}\cos 3\theta\right| \leq \frac{1}{8}$

$\Rightarrow |\cos 3\theta| \leq \frac{1}{2}$

$\Rightarrow -\frac{1}{2} \leq \cos 3\theta \leq \frac{1}{2}$

$\Rightarrow$ maximum value of cos3θ = $\frac{1}{2}$, here

$\Rightarrow 3\theta = 2n\pi \pm \frac{\pi}{3}$

$\theta = \frac{2n\pi}{3} \pm \frac{\pi}{9}$

As θ ∈ [0, 2π] possible values are

$\theta = \left\{\frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}, \frac{11\pi}{9}, \frac{13\pi}{9}, \frac{17\pi}{9}\right\}$

Whose sum is

$\frac{\pi}{9} + \frac{5\pi}{9} + \frac{7\pi}{9} + \frac{11\pi}{9} + \frac{13\pi}{9} + \frac{17\pi}{9} = \frac{54\pi}{9} = 6\pi$

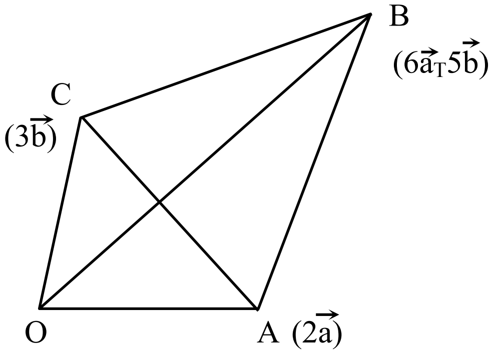
10. Let $\overrightarrow{OA} = 2\vec{a}, \overrightarrow{OB} = 6\vec{a} + 5\vec{b}$ and $\overrightarrow{OC} = 3\vec{b}$, where O is the origin. If the area of the parallelogram with adjacent sides $\overrightarrow{OA}$ and $\overrightarrow{OC}$ is 15 sq. units, then the area (in sq. units) of the quadrilateral OABC is equal to :

(1) 38 (2) 40

(3) 32 (4) 35

Ans. (4)

Sol.



Area of parallelogram having sides

$$\overrightarrow{OA} \,\&\, \overrightarrow{OC} = \left|\overrightarrow{OA} \times \overrightarrow{OC}\right| = \left|2\vec{a} \times 3\vec{b}\right| = 15$$

$$6\left|\vec{a} \times \vec{b}\right| = 15$$

$$\Rightarrow \left|\vec{a} \times \vec{b}\right| = \frac{5}{2} \quad ....(1)$$

Area of quadrilateral

$$OABC = \frac{1}{2}\left|\vec{d}_1 \times \vec{d}_2\right|$$

$$= \frac{1}{2}\left|\overrightarrow{AC} \times \overrightarrow{OB}\right| = \frac{1}{2}\left|\left(3\vec{b} - 2\vec{a}\right) \times \left(6\vec{a} + 5\vec{b}\right)\right|$$

$$= \frac{1}{2}\left|18\vec{b} \times \vec{a} - 10\vec{a} \times \vec{b}\right| = 14\left|\vec{a} \times \vec{b}\right|$$

$$= 14 \times \frac{5}{2} = 35$$

11. If the domain of the function

$$f(x) = \sin^{-1}\left(\frac{x-1}{2x+3}\right) \text{ is } R - (\alpha, \beta)$$

then $12\alpha\beta$ is equal to :

(1) 36 (2) 24

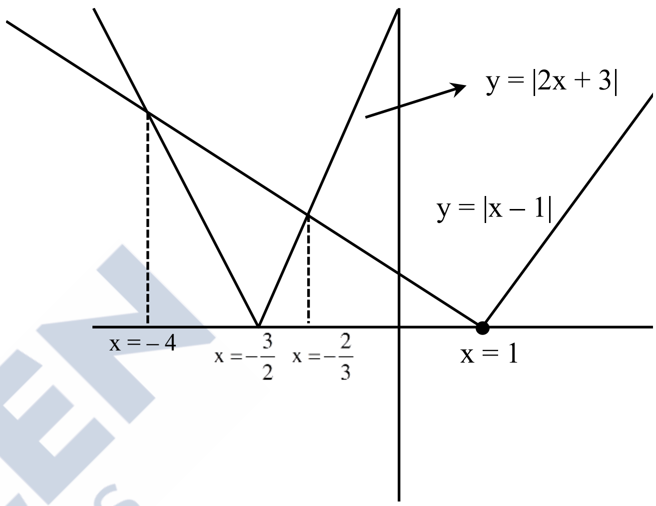
(3) 40 (4) 32

Ans. (4)

Sol. Domain of $f(x) = \sin^{-1}\left(\frac{x-1}{2x+3}\right)$ is

$$2x + 3 \neq 0 \,\&\, x \neq \frac{-3}{2} \text{ and } \left|\frac{(x-1)}{2x+3}\right| \leq 1$$

$$|x-1| \leq |2x+3|$$



For $|2x+3| \geq |x-1|$

$$x \in (-\infty, -4] \cup \left(-\frac{2}{3}, \infty\right)$$

$$\alpha = -4 \,\&\, \beta = -\frac{2}{3} : 12\alpha\beta = 32$$

12. If the sum of series

$$\frac{1}{1\cdot(1+d)} + \frac{1}{(1+d)(1+2d)} + ... + \frac{1}{(1+9d)(1+10d)}$$

is equal to 5, then 50d is equal to :

(1) 20 (2) 5

(3) 15 (4) 10

Ans. (2)

Sol. $$\frac{1}{1\cdot(1+d)} + \frac{1}{(1+d)(1+2d)} + ...$$

$$\frac{1}{(1+9d)(1+10d)} = 5$$

$$\frac{1}{d}\left[\frac{(1+d)-1}{1\cdot(1+d)}+\frac{(1+2d)-(1-d)}{(1+d)(1+2d)}\right]+\ldots\ldots\ldots\ldots$$

$$\frac{(1+10d)-(1+9d)}{(1+9d)(1+10d)}=5$$

$$\frac{1}{d}\left[\left(1-\frac{1}{1+d}\right)+\left(\frac{1}{1+d}-\frac{1}{1+2d}\right)+\ldots\ldots\ldots\ldots\right.$$

$$\left.\left(\frac{1}{1+9d}-\frac{1}{1+10d}\right)\right]=5$$

$$\frac{1}{d}\left[1-\frac{1}{(1+10d)}\right]=5$$

$$\frac{10d}{1+10d}=5d$$

$$50d=5$$

13. Let $f(x)=ax^3+bx^2+ex+41$ be such that $f(1) = 40$, $f'(1) = 2$ and $f''(1) = 4$.

Then $a^2 + b^2 + c^2$ is equal to :

(1) 62 (2) 73

(3) 54 (4) 51

Ans. (4)

Sol. $f(x)=ax^3+bx^2+cx+41$

$f'(x)=3ax^2+2bx+cx$

$\Rightarrow f'(1)=3a+2b+c=2$ .... (1)

$f''(n)=6ax+2b$

$\Rightarrow f''(1)=6a+2b=4$

$3a + b = 2$ .... (2)

(1) – (2)

$b + c = 0$ .... (3)

$f(1) = 40$

$a + b + c + 41 = 40$

use (3)

$a + 41 = 40$

by (2)

$-3 + b = 2 \Rightarrow b = 5$ & $c = -5$

$a^2 + b^2 + c^2 = 1 + 25 + 25 = 51$

14. Let a circle passing through (2, 0) have its centre at the point (h, k). Let (x_c, y_c) be the point of intersection of the lines $3x + 5y = 1$ and $(2 + c)x + 5c^2y = 1$. If $h=\lim_{c\to1}x_c$ and $k=\lim_{c\to1}y_c$, then the equation of the circle is :

(1) $25x^2 + 25y^2 - 20x + 2y - 60 = 0$

(2) $5x^2 + 5y^2 - 4x - 2y - 12 = 0$

(3) $25x^2 + 25y^2 - 2x + 2y - 60 = 0$

(4) $5x^2 + 5y^2 - 4x + 2y - 12 = 0$

Ans. (1)

Sol. $(2+c)x+5c^2\left(\frac{1-3x}{5}\right)=1$

$$x=\frac{1-c^2}{2+c-3c^2},\ y=\frac{1-3x}{5}=\frac{c-1}{5(2+c-3c^2)}$$

$$h=\lim_{c\to1}\frac{(1-c)(1+c)}{(1-c)(2+3c)}=\frac{2}{5}$$

$$K=\lim_{c\to1}\frac{c-1}{-5(c-1)(3c+2)}=-\frac{1}{25}$$

$$\text{Centre}\left(\frac{2}{25},-\frac{1}{25}\right),$$

$$r=\sqrt{\left(2-\frac{2}{5}\right)^2+\left(0-\frac{1}{25}\right)^2}=\sqrt{\frac{64}{25}+\frac{1}{625}}$$

$$r=\frac{\sqrt{161}}{25}$$

$$\left(x-\frac{2}{5}\right)^2+\left(y+\frac{1}{25}\right)^2=\frac{161}{125}$$

$$\Rightarrow 25x^2+25y^2-20x+2y-60=0$$

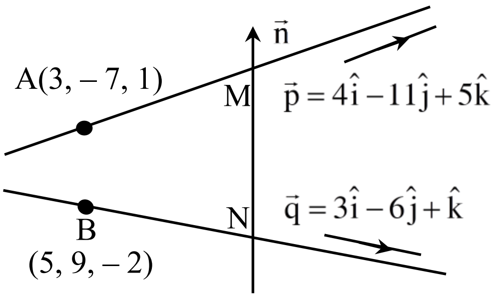
15. The shortest distance between the line $\frac{x-3}{4}=\frac{y+7}{-11}=\frac{z-1}{5}$ and $\frac{x-5}{3}=\frac{y-9}{-6}=\frac{z+2}{1}$ is :

(1) $\frac{187}{\sqrt{563}}$ (2) $\frac{178}{\sqrt{563}}$

(3) $\frac{185}{\sqrt{563}}$ (4) $\frac{179}{\sqrt{563}}$

Ans. (1)

Sol.



$\vec{n} = \vec{p} \times \vec{q}$

$$\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -11 & 5 \\ 3 & -6 & 1 \end{vmatrix} = 19\hat{i} + 11\hat{j} + 9\hat{k}$$

S.d. = projection of $\overrightarrow{AB}$ on $\vec{n}$

$$= \left|\frac{\overrightarrow{AB}\cdot\vec{n}}{|\vec{n}|}\right| = \left|\frac{(2\hat{i} + 16\hat{j} - 3\hat{k}).(19\hat{i} + 11\hat{j} + 9\hat{k})}{\sqrt{361 + 121 + 81}}\right|$$

$$= \frac{38 + 176 - 27}{\sqrt{563}}$$

$$\text{S.d.} = \frac{187}{\sqrt{563}}$$

16. The frequency distribution of the age of students in a class of 40 students is given below.

Age	15	16	17	18	19	20
No. of Students	5	8	5	12	x	y

If the mean deviation about the median is 1.25, then 4x + 5y is equal to :

(1) 43 (2) 44

(3) 47 (4) 46

Ans. (2)

Sol. x + y = 10 ....(1)

Median = 18 = M

$$\text{M.D.} = \frac{\sum f_i |x_i - M|}{\sum f_i}$$

$$1.25 = \frac{36 + x + 2y}{40}$$

x + 2y = 14 ....(1)

by (1) & (2)

x = 6, y = 4

⇒ 4x + 5y = 24 + 20 = 44

Age(x_i)	f	$\lvert x_i - M\rvert$	$f_i\lvert x_i - M\rvert$
15	5	3	15
16	8	2	16
17	5	1	5
18	12	0	0
19	x	1	x
20	y	2	2y

17. The solution of the differential equation $(x^2 + y^2)dx - 5xy\,dy = 0$, y(1) = 0, is :

(1) $\left|x^2 - 4y^2\right|^5 = x^2$ (2) $\left|x^2 - 2y^2\right|^6 = x$

(3) $\left|x^2 - 4y^2\right|^6 = x$ (4) $\left|x^2 - 2y^2\right|^5 = x^2$

Ans. (1)

Sol. $(x^2 + y^2)\,dx = 5xydy$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2 + y^2}{5xy}$$

Put $y = Vx$

$$\Rightarrow V + x\frac{dv}{dx} = \frac{1 + V^2}{5V}$$

$$\Rightarrow \frac{xdv}{dx} = \frac{1 - 4V^2}{5V}$$

$$\Rightarrow \int \frac{V}{1 - 4V^2}dV = \int \frac{dx}{5x}$$

Let $1 - 4V^2 = t$

⇒ – 8V dV = dt

$\Rightarrow \int \frac{dt}{(-8)(t)} = \int \frac{dx}{5x}$

$\Rightarrow \frac{-1}{8} \ln|t| = \frac{1}{5} \ln|x| + \ln C$

$\Rightarrow -5\ln|t| = 8\ln|x| + \ln K$

$\Rightarrow \ln x^8 + \ln|t^5| + \ln K = 0$

$\Rightarrow x^8 |t^5| = C$

$\Rightarrow x^8 |1-4V^2|^5 = C$

$\Rightarrow x^8 \left|\frac{x^2-4y^2}{x^2}\right|^5 = C$

$\Rightarrow |x^2-4y^2|^5 = Cx^2$

given y(1) = 0

$\Rightarrow |1|^5 = C \Rightarrow C = 1$

$\Rightarrow |x^2-4y^2|^5 = x^2$

18. Let three vectors $\vec{a} = \alpha\hat{i} + 4\hat{j} + 2\hat{k}$, $\vec{b} = 5\hat{i} + 3\hat{j} + 4\hat{k}$, $\vec{c} = x\hat{i} + y\hat{j} + z\hat{k}$ from a triangle such that $\vec{c} = \vec{a} - \vec{b}$ and the area of the triangle is $5\sqrt{6}$. if α is a positive real number, then $|\vec{c}|^2$ is :

(1) 16 (2) 14

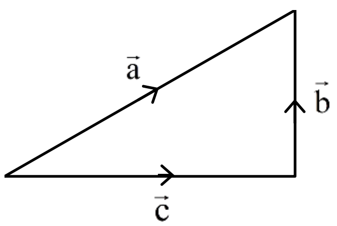
(3) 12 (4) 10

Ans. (2)

Sol. $\vec{c} = \vec{a} - \vec{b}$

$\Rightarrow$ (x, y, z) = (α–5, 1, –2)

$\Rightarrow$ x = α–5, y = 1, z = – 2 ....(1)



Area of Δ = $5\sqrt{6}$ (given)

$\frac{1}{2}|\vec{a} \times \vec{c}| = 5\sqrt{6}$

$\left\| \begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ \alpha & 4 & 2 \\ x & 1 & -2 \end{matrix} \right\| = 10\sqrt{6}$

$\Rightarrow |-10\hat{i} - \hat{j}(-2\alpha - 2x) + \hat{k}(\alpha - 4x)| = 10\sqrt{6}$

$\Rightarrow (2\alpha + 2\alpha - 10)^2 + (\alpha - 4\alpha + 20)^2 = 500$

$\Rightarrow (4\alpha - 10)^2 + (20 - 3\alpha)^2 = 500$

$\Rightarrow 25\alpha^2 - 80\alpha - 120\alpha = 0$

$\Rightarrow \alpha(25\alpha - 200) = 0$

$\Rightarrow$ α = 8 (given α is +ve number)

$\Rightarrow$ x = α – 5 = 3

$|\vec{c}|^2 = x^2 + y^2 + z^2$

= 9 + 1 + 4

= 14

19. Let α, β be the roots of the equation $x^2 + 2\sqrt{2}x - 1 = 0$. The quadratic equation, whose roots are $\alpha^4 + \beta^4$ and $\frac{1}{10}(\alpha^6 + \beta^6)$, is :

(1) $x^2 - 190x + 9466 = 0$

(2) $x^2 - 195x + 9466 = 0$

(3) $x^2 - 195x + 9506 = 0$

(4) $x^2 - 180x + 9506 = 0$

Ans. (3)

Sol. $x^2 + 2\sqrt{2}x - 1 = 0$

$\alpha + \beta = -2\sqrt{2}$

$\alpha\beta = -1$

$\alpha^4 + \beta^4 = (\alpha^2 + \beta^2)^2 - 2\alpha^2\beta^2$

$= ((\alpha + \beta)^2 - 2\alpha\beta)^2 - 2(\alpha\beta)^2$

$= (8 + 2)^2 - 2(-1)^2$

$= 100 - 2 = 98$

$\alpha^6 + \beta^6 = (\alpha^3 + \beta^3)^2 - 2\alpha^3\beta^3$

$= ((\alpha + \beta)((\alpha + \beta)^2 - 3\alpha\beta)^2 - 2(\alpha\beta)^3$

$= (-2\sqrt{2}\,(8 + 3))^2 + 2$

$= (8)\,(121) + 2 = 970$

$\frac{1}{10}\,(\alpha^6 + \beta^6) = 97$

$x^2 - (98 + 97)x + (98)\,(97) = 0$

$\Rightarrow x^2 - 195x + 9506 = 0$

20. Let $f(x) = x^2 + 9$, $g(x) = \frac{x}{x-9}$ and a = fog(10), b = gof(3). If e and l denote the eccentricity and the length of the latus rectum of the ellipse $\frac{x^2}{a} + \frac{y^2}{b} = 1$, then $8e^2 + l^2$ is equal to.

(1) 16 (2) 8

(3) 6 (4) 12

Ans. (2)

Sol. $f(x) = x^2 + 9 \quad g(x) = \frac{x}{x-9}$

$a = f(g(10)) = f\left(\frac{10}{10-9}\right)$

$= f(10) = 109$

$b = g\,(f(3)) = g\,(9 + 9)$

$= g(18) = \frac{18}{9} = 2$

$E : \frac{x^2}{109} + \frac{y^2}{2} = 1$

$e^2 = 1 - \frac{2}{109} = \frac{107}{109}$

$\ell = \frac{2(2)}{\sqrt{109}} = \frac{4}{\sqrt{109}}$

$8e^2 + \ell^2 = \frac{8(107)}{109} + \frac{16}{109}$

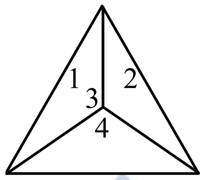
$= 8$

SECTION-B

21. Let a, b and c denote the outcome of three independent rolls of a fair tetrahedral die, whose four faces are marked 1, 2, 3, 4. If the probability that $ax^2 + bx + c = 0$ has all real roots is $\frac{m}{n}$, gcd(m, n) = 1, then m + n is equal to ________.

Ans. (19)

Sol. $a, b, c \in \{1, 2, 3, 4\}$



Tetrahedral dice

$ax^2 + bx + c = 0$

has all real roots

$\Rightarrow D \geq 0$

$\Rightarrow b^2 - 4ac \geq 0$

Let $b = 1 \Rightarrow 1 - 4ac \geq 0$ (Not feasible)

$b = 2 \Rightarrow 4 - 4ac \geq 0$

$1 \geq ac \Rightarrow a = 1, c = 1,$

$b = 3 \Rightarrow 9 - 4ac \geq 0$

$\frac{9}{4} \geq ac$

$\Rightarrow a = 1, c = 1$

$\Rightarrow a = 1, c = 2$

$\Rightarrow a = 2, c = 1$

$b = 4 \Rightarrow 16 - 4ac \geq 0$

$4 \geq ac$

$\Rightarrow a = 1, c = 1$

$\Rightarrow a = 1, c = 2 \quad \Rightarrow a = 2, c = 1$

$\Rightarrow a = 1, c = 3 \quad \Rightarrow a = 3, c = 1$

$\Rightarrow a = 1, c = 4 \quad \Rightarrow a = 4, c = 1$

$\Rightarrow a = 2, c = 2$

Probability $= \frac{12}{(4)(4)(4)} = \frac{3}{16} = \frac{m}{m}$

$m + n = 19$

22. The sum of the square of the modulus of the elements in the set $\{z = a + ib : a, b \in Z, z \in C, |z-1| \le 1, |z-5| \le |z-5i|\}$ is ________.

Ans. (9)

Sol. $|z-1| \le 1$

$\Rightarrow |(x-1)+iy| \le 1$

$\Rightarrow \sqrt{(x-1)^2 + y^2} \le 1$

$\Rightarrow (x-1)^2 + y^2 \le 1$ .... (1)

Also $|z-5| \le |z-5i|$

$(x-5)^2 + y^2 \le x^2 + (y-5)^2$

$-10x \le -10y$

$\Rightarrow x \ge y$ .... (2)

Solving (1) and (2)

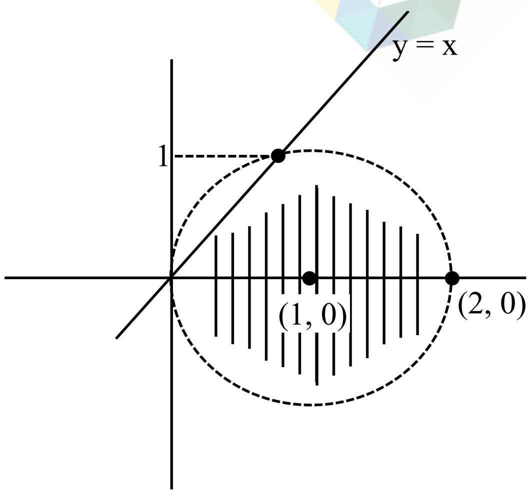
$\Rightarrow (x-1)^2 + x^2 = 1$

$\Rightarrow 2x^2 - 2x = 0$

$\Rightarrow x(x-1) = 0$

$\Rightarrow x = 0 \text{ or } x = 1$

$y = 0 \text{ or } y = 1$



Given x, y ∈ I

Points (0, 0), (1, 0), (2, 0), (1, 1), (1, –1)

to find

$|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 + |z_5|^2$

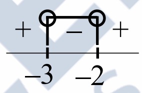
$= 0 + 1 + 4 + 1 + 1 + 1 + 1 = 9$

23. Let the set of all positive values of λ, for which the point of local minimum of the function $(1 + x(\lambda^2 - x^2))$ satisfies $\frac{x^2 + x + 2}{x^2 + 5x + 6} < 0$, be (α, β). Then $\alpha^2 + \beta^2$ is equal to ________.

Ans. (39)

Sol. $\frac{x^2 + x + 2}{x^2 + 5x + 6} < 0$

$\Rightarrow \frac{1}{(x+2)(x+3)} < 0$



$x \in (-3, -2)$ ....(1)

$f(x) = 1 + x(\lambda^2 - x^2)$

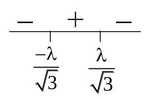
Finding local minima

$f'(x) = (\lambda^2 - x^2) + (-2x).x$

Put f′(x) = 0

$\Rightarrow \lambda^2 = 3x^2$

$\Rightarrow x = \pm\frac{\lambda}{\sqrt{3}}$



Local min Local max

We want local min

$\Rightarrow x = \frac{-\lambda}{\sqrt{3}}$

from (1)

$x \in (-3, -2)$

$-3 < \frac{-\lambda}{\sqrt{3}} < -2$

$3\sqrt{3} > \lambda > 2\sqrt{3}$

$\alpha = 2\sqrt{3}, \beta = 3\sqrt{3}$

$\alpha^2 + \beta^2 = 12 + 27 = 39$

24. Let

$$\lim_{n\to\infty}\left(\frac{n}{\sqrt{n^4+1}} - \frac{2n}{(n^2+1)\sqrt{n^4+1}} + \frac{n}{\sqrt{n^4+16}} - \frac{8n}{(n^2+4)\sqrt{n^4+16}} + \ldots\ldots + \frac{n}{\sqrt{n^4+n^4}} - \frac{2n\cdot n^2}{(n^2+n^2)\sqrt{n^4+n^4}}\right) \text{ be } \frac{\pi}{k},$$

using only the principal values of the inverse trigonometric functions. Then k^2 is equal to _____.

Ans. (32)

Sol. $\sum_{r=1}^{\infty} \frac{n}{\sqrt{n^4+r^4}} - \frac{2nr^2}{(n^2+r^2)\sqrt{n^4+r^4}}$

$$\sum_{r=1}^{\infty} \frac{\frac{1}{n}}{\sqrt{1+\left(\frac{r}{n}\right)^4}} - \frac{2\left(\frac{1}{n}\right)\left(\frac{r}{n}\right)^2}{\left(1+\left(\frac{r}{n}\right)^2\right)\sqrt{1+\left(\frac{r}{n}\right)^4}}$$

$$\Rightarrow \int_0^1 \frac{dx}{\sqrt{1+x^4}} - \frac{2x^2 dx}{(1+x^2)\sqrt{1+x^4}}$$

$$\Rightarrow \int_0^1 \frac{1-x^2}{(1+x^2)\sqrt{1+x^4}} dx$$

$$\Rightarrow \int_0^1 \frac{\frac{1}{x^2}-1}{\left(x+\frac{1}{x}\right)\sqrt{x^2+\frac{1}{x^2}}} dx$$

$$\Rightarrow -\int_0^1 \frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)\sqrt{\left(x+\frac{1}{x}\right)^2-2}} dx$$

$x+\frac{1}{x} = t \Rightarrow 1-\frac{1}{x^2}dx = dt$

$$\Rightarrow -\int_{\infty}^{2} \frac{dt}{t\sqrt{t^2-2}}$$

$$\Rightarrow -\int_{\infty}^{2} \frac{tdt}{t^2\sqrt{t^2-2}}$$

take $t^2 - 2 = \alpha^2$

$t\,dt = \alpha\, d\alpha$

$$\Rightarrow -\int_{\infty}^{\sqrt{2}} \frac{\alpha d\alpha}{(\alpha^2+2)\alpha}$$

$$\Rightarrow -\int_{\infty}^{\sqrt{2}} \frac{d\alpha}{\alpha^2+2}$$

$$\Rightarrow \frac{-1}{\sqrt{2}} \tan^{-1}\frac{\alpha}{\sqrt{2}}\Big]_{\infty}^{\sqrt{2}}$$

$$\Rightarrow \frac{-1}{\sqrt{2}}\{\tan^{-1}1\} + \frac{1}{\sqrt{2}}\tan^{-1}\infty$$

$$\Rightarrow \frac{1}{\sqrt{2}}\left\{\frac{\pi}{2}-\frac{\pi}{4}\right\}$$

$$\Rightarrow \frac{\pi}{4\sqrt{2}} = \frac{\pi}{K}$$

So $K = 4\sqrt{2}$

$K^2 = 32$

25. The remainder when 428^{2024} is divided by 21 is _______.

Ans. (1)

Sol. $(428)^{2024} = (420 + 8)^{2024}$

$= (21 \times 20 + 8)^{2024}$

$= 21m + 8^{2024}$

Now $8^{2024} = (8^2)^{1012}$

$= (64)^{1012}$

$= (63 + 1)^{1012}$

$= (21 \times 3 + 1)^{1012}$

$= 21n + 1$

$\Rightarrow$ Remainder is 1.

26. Lef f:(0, π) → R be a function given by

$$f(x)=\begin{cases}\left(\frac{8}{7}\right)^{\frac{\tan 8x}{\tan 7x}}, & 0<x<\frac{\pi}{2}\\ a-8, & x=\frac{\pi}{2}\\ (1+|\cot x|)^{\frac{b}{a}|\tan x|}, & \frac{\pi}{2}<x<\pi\end{cases}$$

Where a, b ∈ Z. If f is continuous at $x=\frac{\pi}{2}$, then a^2+b^2 is equal to ________.

Ans. (81)

Sol. LHL at $x=\frac{\pi}{2}$

$$\lim_{x\to\frac{\pi}{2}}\left(\frac{8}{7}\right)^{\frac{\tan 8x}{\tan 7x}}=\left(\frac{8}{7}\right)^{0}=1$$

RHL at $x=\frac{\pi}{2}$

$$\lim_{x\to\frac{\pi}{2}}(1+|\cot x|)^{\frac{b}{a}|\tan x|}$$

$$=e^{\lim\limits_{x\to\frac{\pi}{2}}|\cot x|\frac{b}{a}|\tan x|}=e^{\frac{b}{a}}$$

$\Rightarrow 1=a-8=e^{\frac{b}{a}}$

$\Rightarrow a=9, b=0$

$\Rightarrow a^2+b^2=81$

27. Let A be a non-singular matrix of order 3. If $\det(3\text{adj}(2\text{adj}((\det A)A)))=3^{-13}\cdot 2^{-10}$ and $\det(3\text{adj}(2A))=2^m\cdot 3^n$, then $|3m+2n|$ is equal to ________.

Ans. (14)

Sol. $|3\text{ adj}(2\text{adj }(|A|A))|=|3\text{adj }(2|A|^2\text{ adj}(A)|$

$=|3.2^2|A|^4\text{ adj(adj }(A)|=2^6 3^3\,|A|^{12}\,|A|^4$

$=2^6\,3^3\,|A|^{16}=2^{-10}\,3^{-13}$

$\Rightarrow |A|^{16}=2^{-16}\,3^{-16}\ \Rightarrow |A|=2^{-1}\,3^{-1}$

Now $|3\text{adj }(2A)|=|3.2^2\text{ adj}(A)|$

$=2^6\,3^3\,|A|^2=2^{-m}\,3^{-n}$

$\Rightarrow 2^6\,3^3\,2^{-2}\,3^{-2}=2^{-m}\,3^{-n}$

$\Rightarrow 2^{-m}\,3^{-n}=2^4\,3^1$

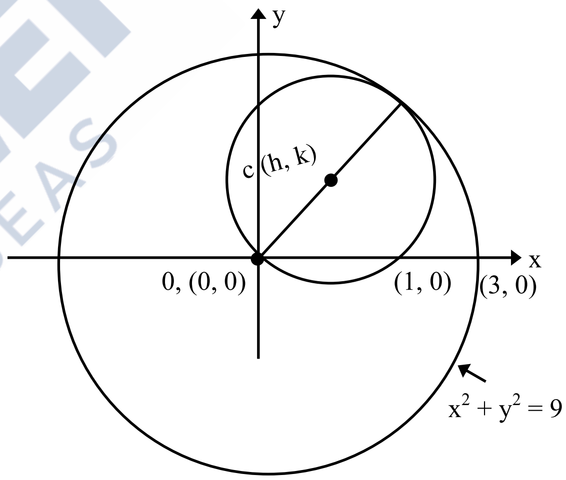
$\Rightarrow m=-4, n=-1$

$\Rightarrow |3m+2n|=|-12-2|=14$

28. Let the centre of a circle, passing through the point (0, 0), (1, 0) and touching the circle $x^2+y^2=9$, be (h, k). Then for all possible values of the coordinates of the centre (h, k), $4(h^2+k^2)$ is equal to ________.

Ans. (9)

Sol.



$(x-h)^2+(y-k)^2=h^2+k^2$

$x^2+y^2-2hx-2ky=0$

∵ passes through (1, 0)

$\Rightarrow 1+0-2h=0$

$\Rightarrow h=1/2$

$\because OC=\frac{OP}{2}$

$$\sqrt{\left(\frac{1}{2}\right)^2+k^2}=\frac{3}{2}$$

$$\frac{1}{4}+k^2=\frac{9}{4}$$

$k^2 = 2$

$k = \pm\sqrt{2}$

$\therefore$ Possible coordinate of

$$c\,(h, k)\left(\frac{1}{2},\sqrt{2}\right)\left(\frac{1}{2},-\sqrt{2}\right)$$

$$4(h^2+k^2)=4\left(\frac{1}{4}+2\right)=4\left(\frac{9}{4}\right)=9$$

29. If a function f satisfies f(m + n) = f(m) + f(n) for all m, n $\in$ N and f(1) = 1, then the largest natural number λ such that $\sum_{k=1}^{2022} f(\lambda+k)\le(2022)^2$ is equal to ________.

Ans. (1010)

Sol. $f(m+n) = f(m) + f(n)$

$\Rightarrow f(x) = kx$

$\Rightarrow f(1) = 1$

$\Rightarrow k = 1$

$f(x) = x$

Now

$$\sum_{k=1}^{2022} f(\lambda+k)\le(2022)^2$$

$$\Rightarrow \sum_{k=1}^{2022}(\lambda+k)\le(2022)^2$$

$$\Rightarrow 2022\lambda+\frac{2022\times2023}{2}\le(2022)^2$$

$$\Rightarrow \lambda\le 2022-\frac{2023}{2}$$

$$\Rightarrow \lambda\le 1010.5$$

$\therefore$ largest natural no. λ is 1010.

30. Let A = {2, 3, 6, 7} and B = {4, 5, 6, 8}. Let R be a relation defined on A × B by (a_1, b_1) R (a_2, b_2) is and only if $a_1 + a_2 = b_1 + b_2$. Then the number of elements in R is ________.

Ans. (25)

Sol. A = {2, 3, 6, 7}

B = { 2, 5, 6, 8}

(a_1, b_1) R (a_2, b_2)

$a_1 + a_2 = b_1 + b_2$

1. (2, 4) R (6, 4)	2. (2, 4) R (7, 5)	× 2
3. (2,5) R (7, 4)	4. (3, 4) R (6, 5)	
5. (3, 5) R (6, 4)	6. (3, 5) R (7, 5)	
7. (3,6) R (7, 4)	8. (3, 4) R (7, 6)	
9. (6, 5) R (7, 8)	10. (6, 8) R (7, 5)	
11. (7,8) R (7, 6)	12. (6, 8) R (6, 4)	
13. (6,6) R (6, 6)		

Total 24 + 1 = 25